SCOTTISH QUALIFICATIONS AUTHORITY

2002 Mathematics

Advanced Higher

Finalised Marking Instructions

SECTION A (Mathematics 1 and 2)

	Solution	Marks
A1.	$\begin{array}{cccc} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 5 & 5 \end{array} \Rightarrow \begin{array}{cccc} 1 & 1 & 3 & 2 \\ & -1 & -5 & -2 \\ & -1 & -4 & -1 \end{array}$	Second row 1 mark Third row 1 mark
	$\Rightarrow \begin{array}{cccc} 1 & 1 & 3 & 2 \\ & -1 & -5 & -2 \\ & & -1 & -1 \end{array}$	Third row 1 mark
	$z = 1; \quad y = -3; \quad x = 2$	Values 2E1. (available whatever method used above) Total 5
A2.	$i^4 + 4i^3 + 3i^2 + 4i + 2$ $= 1 - 4i - 3 + 4i + 2 = 0$	1 mark for verifying and stating
	Since i is a root, $-i$ must also be a root. Thus factors $(z - i)$ and $(z + i)$ give a quadratic factor $z^2 + 1$.	1 for getting $-i$. 1 for $z^2 + 1$ is a factor.
	$\begin{array}{r} z^2 + 4z + 2 \\ z^2+1 \,\overline{)\, z^4 + 4z^3 + 3z^2 + 4z + 2} \\ z^4 \quad\quad + z^2 \quad\quad\quad\quad \\ \hline 4z^3 + 2z^2 + 4z \quad \\ 4z^3 \quad\quad + 4z \quad \\ \hline 2z^2 \quad\quad + 2 \end{array}$	1 for factorisation.
	Solving $z^2 + 4z + 2 = 0$ gives $z = -2 \pm \sqrt{2}$.	1 for the other two roots. Total 5
A3.	At A, $x = -1$ so $t^2 + t - 1 = -1$ giving $t = 0$ or $t = -1$. When $t = 0$, $y = 2$. When $t = -1$, $y = 5$ so A is on the curve.	1 for solving a quadratic. 1 for the other coordinate.
	$\frac{dx}{dt} = 2t + 1; \; \frac{dy}{dt} = 4t - 1$	1 for $\frac{dx}{dt}$ **and** $\frac{dy}{dt}$.
	$\frac{dy}{dx} = \frac{4t - 1}{2t + 1}$.	1 for $\frac{dy}{dx}$.
	When $t = -1$, $\frac{dy}{dx} = \frac{-5}{-1} = 5$.	1 for the gradient is 5.
	The equation is $(y - 5) = 5(x + 1)$ $y = 5x + 10$	1 for an equation. Total 6

A4.

(a) $f(x) = \sqrt{x}e^{-x} = x^{1/2}e^{-x}$

$$f'(x) = \tfrac{1}{2}x^{-1/2}e^{-x} + x^{1/2}(-1)e^{-x}$$

$$= \frac{1}{2\sqrt{x}}e^{-x}(1 - 2x)$$

1 method mark — 1 for first term, 1 for second term

1 for a factorised form

(b) $y = (x + 1)^2(x + 2)^{-4}$

$$\log y = 2\log(x+1) - 4\log(x+2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x+1} - \frac{4}{x+2}$$

$$\frac{dy}{dx} = \left(\frac{2}{x+1} - \frac{4}{x+2}\right)y$$

$a = 2;\ b = -4$

1 for taking logs and expanding

1 for differentiating

A logarithmic approach is needed.

1 for rearranging

Total 7

A5.

$$\int_0^1 \ln(1 + x)\,dx$$

$$= \int_0^1 \ln(1 + x)\,.\,1\,dx$$

1 for introducing the factor of 1

$$= \left[x\ln(1 + x) - \int\frac{1}{1 + x}.x\,dx\right]_0^1$$

1 for second term

$$= \left[x\ln(1 + x) - \int\left(1 - \frac{1}{1 + x}\right)dx\right]_0^1$$

2 marks for correct manipulation and integration of the second term

$$= [x\ln(1 + x) - x + \ln(1 + x)]_0^1$$

$$= [\ln 2 - 1 + \ln 2] - [0 - 0 + 0]$$

1 for limits

$$= 2\ln 2 - 1\,[\approx 0.3863].$$

Total 5

A6.

$$x + 2 = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\,d\theta$$

1 for derivative

Also, $x = 2\tan\theta - 2$, so

$x^2 = 4\tan^2\theta - 8\tan\theta + 4$, giving

$x^2 + 4x + 8 = 4\tan^2\theta + 4$

1 for manipulation

$$\int\frac{dx}{x^2 + 4x + 8} = \int\frac{2\sec^2\theta\,d\theta}{4(\tan^2\theta + 1)}$$

1 for substitution

$$= \frac{1}{2}\int\frac{\sec^2\theta\,d\theta}{\tan^2\theta + 1}$$

$$= \frac{1}{2}\int 1\,d\theta$$

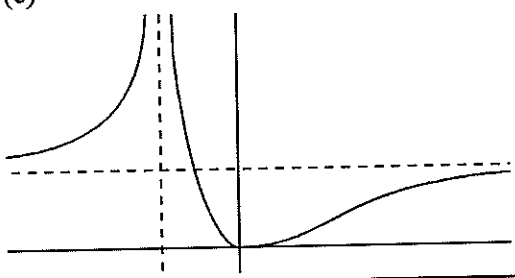
1 for simplifying

$$= \frac{1}{2}\theta + c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x + 2}{2}\right) + c$$

1 for finishing

Total 5

A7.	When $n = 1, 4^n - 1 = 4 - 1 = 3$ so true when $n = 1$. Assume $4^k - 1$ is divisible by 3. Consider $4^{k+1} - 1$. $4^{k+1} - 1 = 4.4^k - 1$ $= (3 + 1)4^k - 1$ $= 3.4^k + (4^k - 1)$ Since both terms are divisible by 3 the result is true for $k + 1$. Thus since true for $n = 1, 4^n - 1$ is divisible by 3 for all $n \geqslant 1$.	1 for the case $n = 1$. 1 for the assumption. 1 for moving to $k + 1$. 1 for a correct formulation. 1 for conclusion. (The involvement of Σ not penalised.) Total 5 Other strategies possible.
A8.	$\frac{x^2}{(x+1)^2} = A + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ so $x^2 = A(x+1)^2 + B(x+1) + C$ $= Ax^2 + (2A+B)x + A + B + C$ Hence $A = 1, B = -2$ and $C = 1$.	1 for valid method 2E1 for the values
	(a) $y = 1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}$ so there is a vertical asymptote $x = -1$ and a horizontal asymptote $y = 1$.	1 for vertical asymptote 1 for horizontal asymptote
	(b) $\frac{dy}{dx} = \frac{2}{(x+1)^2} - \frac{2}{(x+1)^3} = 0$ at SV $\Rightarrow (x+1) = 1 \Rightarrow x = 0, y = 0$ $\frac{d^2y}{dx^2} = \frac{-4}{(x+1)^3} + \frac{6}{(x+1)^4}$ $= -4 + 6$ when $x = 0$ Thus (0, 0) is a minimum.	1 for derivative (however obtained) 1 for solving 1 for justification 1 for (0, 0) is a minimum
	(c) 	1 for asymptotes **or** 1 for each branch 1 for branches Total 11

A9.	(a)	
	$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-xy^2}{-x^2y} = \frac{y}{x}$$	1 mark
	$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$	1 mark
	$$\ln y = \ln x + C$$	1 mark
	$$x = 1, y = 2 \Rightarrow C = \ln 2$$	1 mark for evaluating C
	$$\ln y = \ln x + \ln 2$$	
	$$y = 2x$$	1 mark for formula
	(b)	
	$$\frac{dx}{dt} = -x^2(2x) = -2x^3$$	1 mark
	$$\int \frac{1}{x^3}\,dx = \int -2\,dt$$	1 mark
	$$\frac{x^{-2}}{-2} = -2t + D$$	1 mark
	$$\frac{1}{x^2} = 4t - 2D$$	
	$$t = 0, x = 1 \Rightarrow D = -\frac{1}{2}$$	1 mark
	$$\frac{1}{x^2} = 4t + 1$$	
	$$x = \frac{1}{\sqrt{4t+1}}$$	1 mark Total 10

A10.	$S_n(1) = 1 + 2 + 3 + \dots + n$ $= \frac{1}{2}n(n+1)$	1 for recognising that $S_n(1)$ requires special treatment. 1 for evaluating it correctly.
	$(1-x)S_n(x) = S_n(x) - xS_n(x)$ $= 1 + 2x + 3x^2 + \dots + nx^{n-1}$ $-(x + 2x^2 + 3x^3 + \dots + nx^n)$ $= 1 + x + x^2 + \dots + x^{n-1} - nx^n$	3E1 for expanding correctly and simplifying
	$= \dfrac{1-x^n}{1-x} - nx^n.$ Thus $S_n(x) = \dfrac{1-x^n}{(1-x)^2} - \dfrac{nx^n}{(1-x)}$ as required.	1 for applying the sum of a GP
	$\dfrac{2}{3} + \dfrac{3}{3^2} + \dfrac{4}{3^3} + \dots + \dfrac{n}{3^{n-1}} + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$ $= \left(S_n(\tfrac{1}{3}) - 1\right) + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$	1 for recognising that it relates to $S_n(\frac{1}{3})$.
	$= \dfrac{1-\frac{1}{3^n}}{(1-\frac{1}{3})^2} - \dfrac{n\frac{1}{3^n}}{1-\frac{1}{3}} - 1 + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$ $= \dfrac{9}{4}\left(1 - \dfrac{1}{3^n}\right) - \dfrac{3}{2}\cdot\dfrac{n}{3^n} - 1 + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$ $= \dfrac{5}{4}\left(1 - \dfrac{1}{3^n}\right)$	1 for applying earlier result.
	$\lim_{n\to\infty}\left\{\dfrac{2}{3} + \dfrac{3}{3^2} + \dfrac{4}{3^3} + \dots + \dfrac{n}{3^{n-1}} + \dfrac{3}{2}\cdot\dfrac{n}{3^n}\right\}$ $= \dfrac{5}{4}$	1 for obtaining the limit. Total 9

SECTION B (Mathematics 3)

B1.

(a) $\overrightarrow{AB} = 2\mathbf{i} - \mathbf{k}$; $\overrightarrow{AC} = \mathbf{i} - \mathbf{j} - 3\mathbf{k}$	1 for the two initial vectors
$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 1 & -1 & -3 \end{vmatrix}$	1 for a cross product (Vector form acceptable.)
$= -\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$	1 for the normal vector
Equation of π_1 is of the form $-x + 5y - 2z = c$ $(1,1,0) \Rightarrow c = -1 + 5 = 4$ So an equation is $-x + 5y - 2z = 4$	1 for the equation
(b) Normals are $-\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.	1 for normals
So the angle between the planes is given by $\cos^{-1}\left(\frac{-1 + 10 - 2}{\sqrt{30}\sqrt{6}}\right)$	1 for applying the scalar product
$= \cos^{-1}\frac{7}{6\sqrt{5}}$ [$\approx 58.6°$]	1 for result (must be acute) Total 7

B2.

$A^n = \begin{pmatrix} n + 1 & n \\ -n & 1 - n \end{pmatrix}$. When $n = 1$, RHS $= \begin{pmatrix} 1+1 & 1 \\ -1 & 1-1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = A$. Therefore true when $n = 1$.	1 mark for showing true when $n = 1$
Assume $A^k = \begin{pmatrix} k + 1 & k \\ -k & 1 - k \end{pmatrix}$.	1 for stating the assumption
Consider A^{k+1}. $A^{k+1} = AA^k$	1 for considering $k + 1$
$= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} k + 1 & k \\ -k & 1 - k \end{pmatrix}$ $= \begin{pmatrix} k + 2 & k + 1 \\ -(k + 1) & -k \end{pmatrix}$	1 for this matrix
$= \begin{pmatrix} (k + 1) + 1 & (k + 1) \\ -(k + 1) & 1 - (k + 1) \end{pmatrix}$	1 for obtaining final matrix
Thus if true for k then true for $k + 1$. Since true for $n = 1$, by induction, true for all $n \geqslant 1$.	1 for conclusion Total 6

B3.

$f(x) = \ln(\cos x)$ $f(0) = 0$	1 for first two derivatives
$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$ $f'(0) = 0$	
$f''(x) = -\sec^2 x$ $f''(0) = -1$	
$f'''(x) = -2\sec^2 x \tan x$ $f'''(0) = 0$	1 for third and fourth derivatives
$f''''(x) = -4\sec^3 x \tan^2 x$	1 for evaluation at 0
$-2\sec^4 x$ $f''''(0) = -2$	
$f(x) = f(0) + xf'(0) + \ldots$	1 method mark for series
$\ln(\cos x) = 0 + 0.x - 1.\frac{x^2}{2} + 0.x - 2.\frac{x^4}{4!}$	Using series for log and cos can gain full marks.
$= -\frac{x^2}{2} - \frac{x^4}{12} + \ldots$	1 for an expansion Total 5

B4.

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	1 for A
$B = \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$	1 for B
$BA\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$	1 method for tackling a compostion
$= \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3}x + y \\ x - \sqrt{3}y \end{pmatrix}$	
i.e. $(x,y) \to \frac{1}{2}(\sqrt{3}x + y, x - \sqrt{3}y)$	
so $k = \sqrt{3}$.	1 for value of k Total 4

B5.	$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 4\cos x$	
	A.E. is $m^2 + 2m + 5 = 0$	1 for auxiliary equation
	$\Rightarrow m = -1 \pm 2i$	1 for roots
	C. F. is $y = e^{-x}(A\cos 2x + B\sin 2x)$	1 for form of complementary function
	For P.I. try $f(x) = a\cos x + b\sin x$	
	$f'(x) = -a\sin x + b\cos x$	
	$f''(x) = -a\cos x - b\sin x$	1 for derivatives
	Thus	
	$(4a+2b)\cos x + (4b-2a)\sin x = 4\cos x$	1 for substitution (Use of a wrong PI loses 2 of these 3 marks.)
	$\Rightarrow a = 2b \Rightarrow 10b = 4$	
	$\Rightarrow b = \frac{2}{5}$ and $a = \frac{4}{5}$	1 for values
	$y(x) = e^{-x}(A\cos 2x + B\sin 2x) + \frac{2}{5}(2\cos x + \sin x)$	
	$y(0) = 0 \Rightarrow A + \frac{4}{5} = 0 \Rightarrow A = -\frac{4}{5}$	1 for value of A
	$y'(x) = e^{-x}(-2A\sin 2x + 2B\cos 2x) - e^{-x}(A\cos 2x + B\sin 2x) + \frac{2}{5}(\cos x - 2\sin x)$	1 for derivative
	$y'(0) = 1 \Rightarrow 2B - A + \frac{2}{5} \Rightarrow B = -\frac{1}{10}$	1 for value of B
	$y = \frac{e^{-x}}{10}(-8\cos 2x - \sin 2x) + \frac{2}{5}(2\cos x + \sin x)$	1 for final statement
		Total 10