S4 National 5 Maths Chapter 10

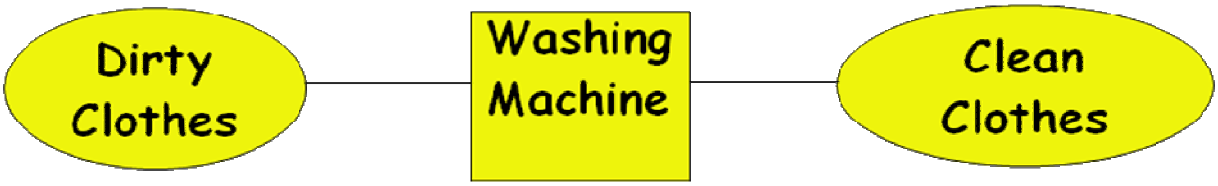
Two men are sitting in the basket of a balloon. For hours, they have been drifting through a thick layer of clouds, and they have lost orientation completely. Suddenly, the clouds part, and the two men see the top of a mountain with a man standing on it.

"Hey! Can you tell us where we are?"
The man doesn't reply. The minutes pass as the balloon drifts past the mountain. When the balloon is about to be swallowed again by the clouds, the man on the mountain shouts: "You're in a balloon!"
"That must have been a mathematician."
"Why?"
"He thought long and thoroughly about what to say,what he eventually said was irrefutably correct, and it was of no use whatsoever."

Functions

Functions

A function describes how one quantity can be related to another.



In this case the first ellipse is the INPUT.
The second ellipse is the OUTPUT.
The middle box is called the FUNCTION.

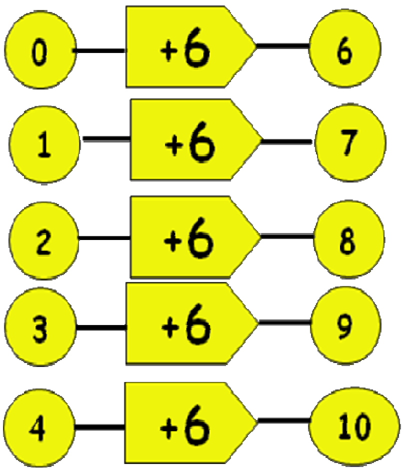
In maths the input is called the DOMAIN. The output is called the RANGE.

Example1

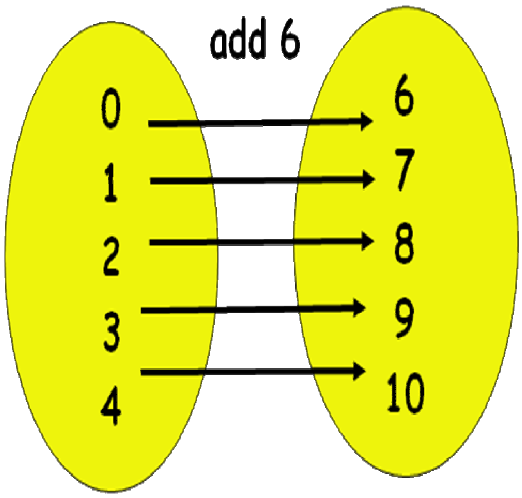
Let f be a mathematical function that means "add 6". The domain is {0, 1, 2, 3, 4} and the range is {6, 7, 8, 9, 10}. Use diagrams to show how f links the domain and the range.

There are three ways of doing this and the question will normally ask you to use one of these representations.

1. Function Machines



2. Arrow Diagram



3. Function Notation

$f : 0 \longrightarrow 6$ This is read as 'f maps 0 onto 6'.

$f : 1 \longrightarrow 7$

$f : 2 \longrightarrow 8$

$f : 3 \longrightarrow 9$

$f : 4 \longrightarrow 10$

A function from a set A to set B is a rule which links each member of A (the domain) to one member of set B (the range).

In general terms we can say that: $f : x \longrightarrow x+6$

This can be written in the form $f(x) = x + 6$.

Ex1

Given $g(x) = x^2 - 3$, find the value of g when:

(a) $x = 1$ (b) $x = 2$

Substitute the x values into the function.

(a) $g(x) = x^2 - 3$

$g(1) = 1^2 - 3$

$= -2$

(b) $g(x) = x^2 - 3$

$g(2) = 2^2 - 3$

$= 1$

Ex2

Given $h(x) = 4x - 1$, for what value of x is $h(x) = -9$.

Set the function equal to -9.

$4x - 1 = -9$

$4x = -8$

$x = -2$

Using The Notation

Ex1

If $f\left(x\right)=4x^{2}$

(a) Find the value if f at x = 3.

(b) Find a formula for:

i. $f\left(t\right)$ ii. $f\left(2t\right)$ iii. $f\left(t-1\right)$

(a)
$$\begin{aligned} f\left(x\right)&=4x^{2}\\ f\left(3\right)&=4\times3^{2}\\ &=4\times9\\ &=36 \end{aligned}$$

(b) i.
$$\begin{aligned} f\left(x\right)&=4x^{2}\\ f\left(t\right)&=4t^{2} \end{aligned}$$

ii.
$$\begin{aligned} f\left(x\right)&=4x^{2}\\ f\left(2t\right)&=4\times\left(2t\right)^{2}\\ &=4\times4t^{2}\\ &=16t^{2} \end{aligned}$$

iii.
$$\begin{aligned} f\left(x\right)&=4x^{2}\\ f\left(t-1\right)&=4\times\left(t-1\right)^{2}\\ &=4\times\left(t^{2}-2t+1\right)\\ &=4t^{2}-8t+4 \end{aligned}$$

Ex2

If $g\left(x\right)=2x-3$, find a formula for $g\left(x\right)+g\left(-k\right)$.

$$\begin{aligned} g\left(x\right)+g\left(-k\right)&=2k-3+\left(2\left(-k\right)-3\right)\\ &=2k-3-2k-3\\ &=-6 \end{aligned}$$

Ex3

If $f:x\longrightarrow3x-1$

(a) Find a formula for $f\left(a\right)$.

(b) Find the value of f at:

i. 0 ii. 1

(a) $f\left(a\right)=3a-1$

(b) i.
$$\begin{aligned} f\left(0\right)&=3\times0-1\\ &=-1 \end{aligned}$$

ii.
$$\begin{aligned} f\left(1\right)&=3\times1-1\\ &=2 \end{aligned}$$

Graphs of Functions

A graph gives a picture of a function.

Linear Functions

A linear function is of the form $f\left(x\right)=ax+b$. Its graph is a straight line with equation $y=ax+b$.

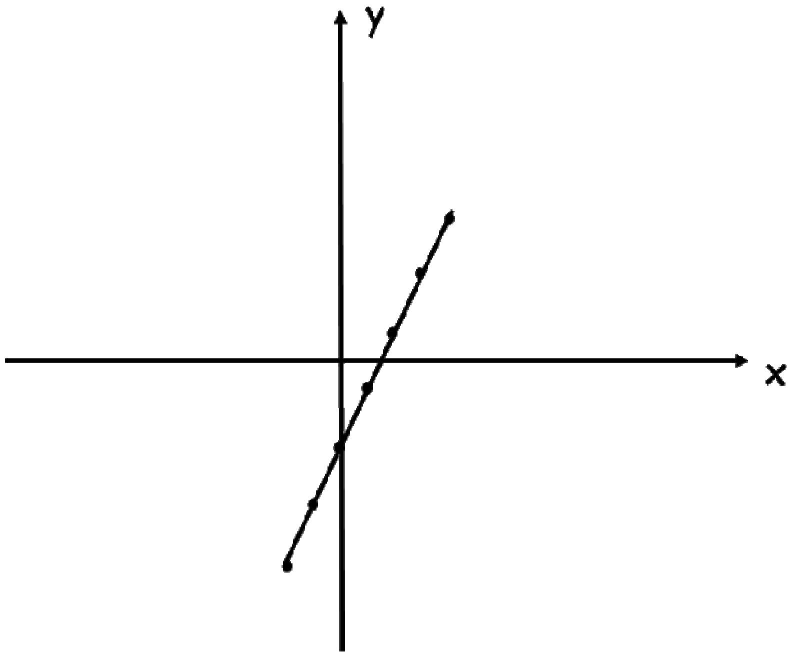
In order to draw the graph of a linear function we need to find at least two points.

Ex1

Draw the graph of the function $f\left(x\right)=2x-3$ for $-2\le x\le 4$.

Set up a table of values that will tell you what the y value of each corresponding x value is.

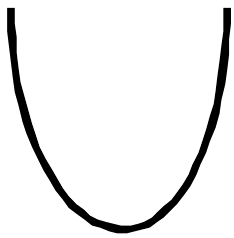
x	-2	-1	0	1	2	3	4
y	-7	-5	-3	-1	1	3	5



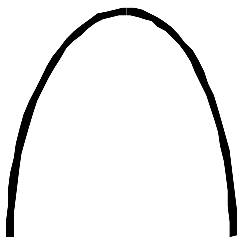
Notice we could have drawn this by just finding two points and drawing the line through them.

Quadratic Functions

A quadratic function is one of the form $f\left(x\right)=ax^2+bx+c$. The shape formed when we draw this is a parabola.



Or



There are several things we can get from the graph of a quadratic:

1. The minimum or maximum value of a quadratic function. This is the y coordinate of the lowest or highest point on the curve.
2. The minimum turning point or maximum turning point. This is the coordinates of the lowest or highest point on the curve.
3. The zeros (roots) of the curve these occur when $f\left(x\right)=0$ i.e. where the curve crosses the x axis and are written in the form x =…
4. The axis of symmetry is a line which splits the graph into two equal parts passing through the turning point and is written in the form x=…

Ex2

(a) Draw the graph of $y=x^2-1$, for $-4\le x\le 4$.
(b) State the minimum value of the function.
(c) Write down the coordinates of the minimum turning point.
(d) The value of x where the function = 0.
(e) The equation of the axis of symmetry for the parabola.

(a) Start with a table of values for x and y.

x	-4	-3	-2	-1	0	1	2	3	4
y	15	8	3	0	-1	0	3	8	15

When $x=-4$ $\quad y=-4^2-1$
$\quad\quad =15$

When $x=-3$ $\quad y=-3^2-1$
$\quad\quad =8$

When $x=-2$ $\quad y=-2^2-1$
$\quad\quad =3$

When $x=-1$ $\quad y=-1^2-1$
$\quad\quad =0$

When $x=0$ $\quad y=0^2-1$
$\quad\quad =-1$

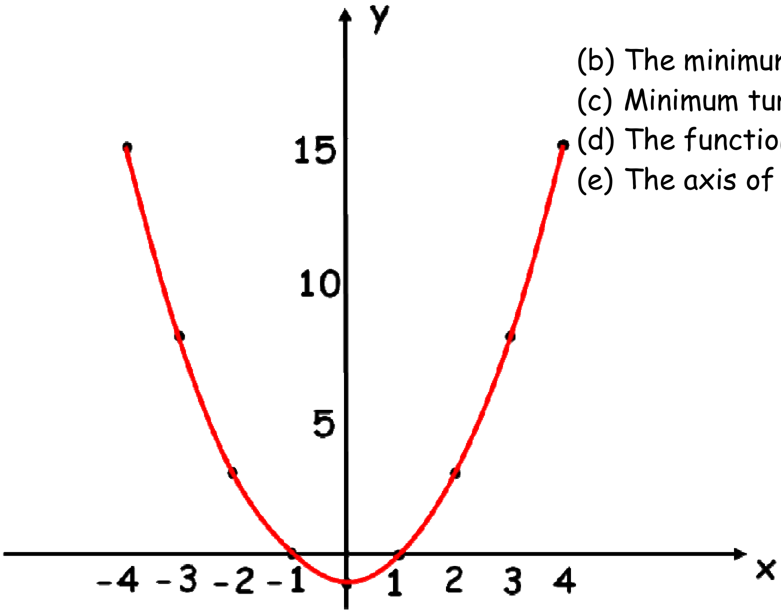
When $x=1$ $\quad y=1^2-1$
$\quad\quad =0$

When $x=2$ $\quad y=2^2-1$
$\quad\quad =3$

When $x=3$ $\quad y=3^2-1$
$\quad\quad =8$

When $x=4$ $\quad y=4^2-1$
$\quad\quad =15$

Now you have all the points, draw a set of axes and plot them.



(b) The minimum value of the function is -1, at $x = 0$
(c) Minimum turning point (tpt) at (0, -1)
(d) The function equals zero at $x = 1$ and $x = -1$
(e) The axis of symmetry of the parabola is the line $x = 0$

<u>Sketching Quadratic Functions</u>

To draw a parabola we have so far used a table. This is not always best since we have to be given the values to draw between. The following method is what we will use from now on to sketch quadratics.

<u>Ex1</u>

Sketch $f(x) = x^2 + 2x - 8$.

<u>Step 1 – Find where the parabola crosses the x-axis:</u>

The parabola will cross the x-axis when y = 0. To do this we need to set the quadratic equal to zero and solve it just like we did in the quadratics chapter.

$$
\begin{aligned}
&x^2 + 2x - 8 = 0 \\
&(x-2)(x+4) = 0 \\
&x - 2 = 0 \text{ or } x + 4 = 0 \\
&x = 2 \text{ or } x = -4 \\
&(2,0) \text{ and } (-4,0)
\end{aligned}
$$

Step 2 – Find the axis of symmetry:
Since parabolas are symmetrical the axis of symmetry is halfway between the points where -the graph crosses x-axis.

To find the point halfway between to x coordinates simply add them together and divide by two.

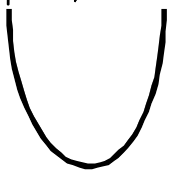
Halfway between x = - 4 and x = 2:
$$x=\frac{-4+2}{2}$$
$$x=\frac{-2}{2}$$
$$x=-1$$

Step 3 - Find the turning point and determine whether it is a maximum or minimum:
Turning point occurs on the axis of symmetry:

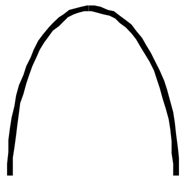
When x = -1
$$\begin{aligned} y &= x^2+2x-8 \\ &= (-1)^2+2(-1)-8 \\ &= 1-2-8 \\ &= -9 \\ &(-1,-9) \end{aligned}$$

To determine whether it is a maximum or minimum turning point, look at the x^2 term.

Positive x^2 means a minimum turning point, or 'happy face'.



Negative x^2 means a maximum turning point, or 'sad face'.

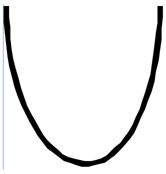


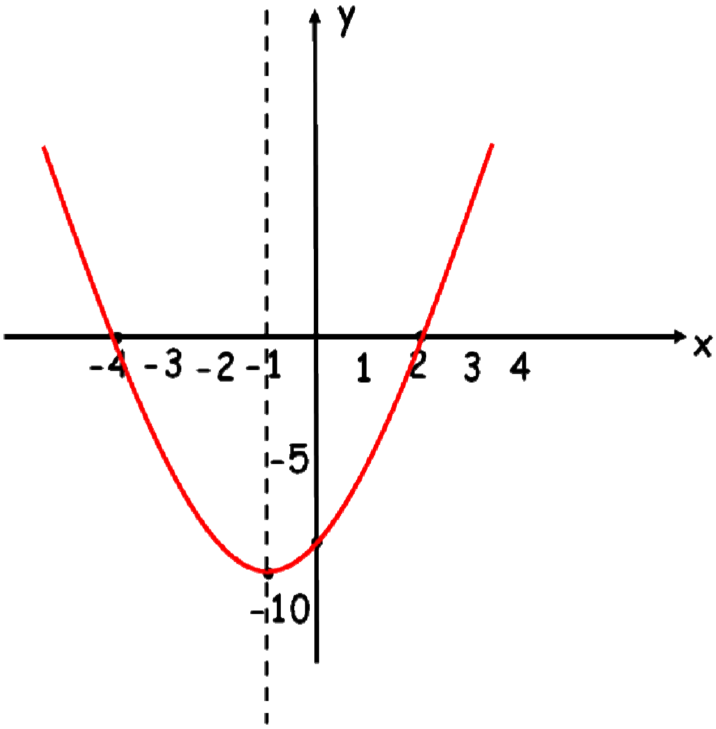
Our function has a positive x^2 term, therefore the turning point is a minimum (happy face!)

Step 4 – Find where the curve cuts the y-axis:
The curve will cut the y-axis when x = 0.

let $x=0$
$$y=0^2+2(0)-8$$
$$y=-8$$
$$(0,-8)$$

<u>Step 5 – Sketch the parabola:</u>
At this stage it is useful to go back through your workings and summarise your information.

Shape - 



Cross x-axis: (-4, 0) & (2, 0)
Equation of line of symmetry: $x = -1$
Minimum turning point at (-1, -9)
Crosses y-axis at (0, -8).

Try this one – Use square paper to sketch the function $15 - 2x - x^2$. Follow each step listed above!

<u>Reciprocal Functions</u>
These are functions where the x is on the bottom of the function.

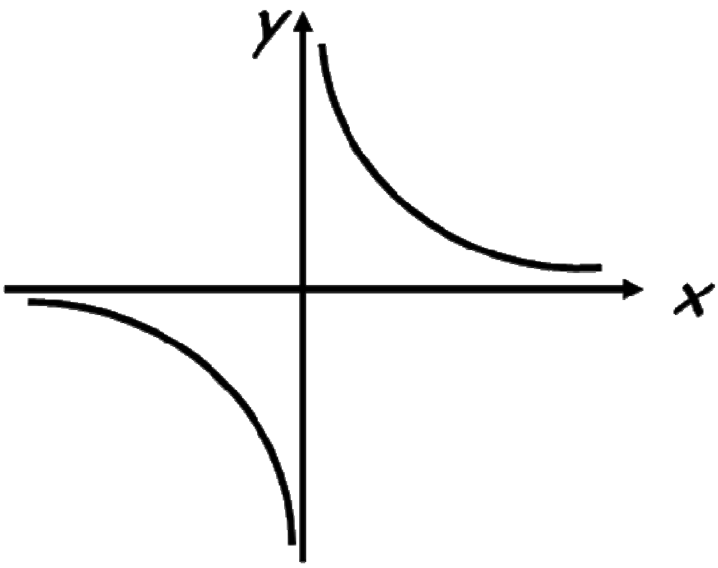
<u>Ex1</u>

Sketch $y = \dfrac{24}{x}$

Start with a table of values:

x	-24	-12	-8	-6	-4	-3	-2	-1	0	1	2	3	4	6	8	12	24
y	-1	-2	-3	-4	-6	-8	-12	-24	-	24	12	8	6	4	3	2	1

If you were to plot this the shape you would see is:



This shape is called a hyperbola and is divided into two branches one for $x < 0$ and one for $x > 0$.

The x and y axis are called asymptotes to the curve since the curve gets closer and closer to them but never meets them.

The curve also has two axis of symmetry $y = x$ and $y = -x$

The curve also has half turn symmetry about the origin.

Exponential Functions

These are functions where the x is the power of the function.

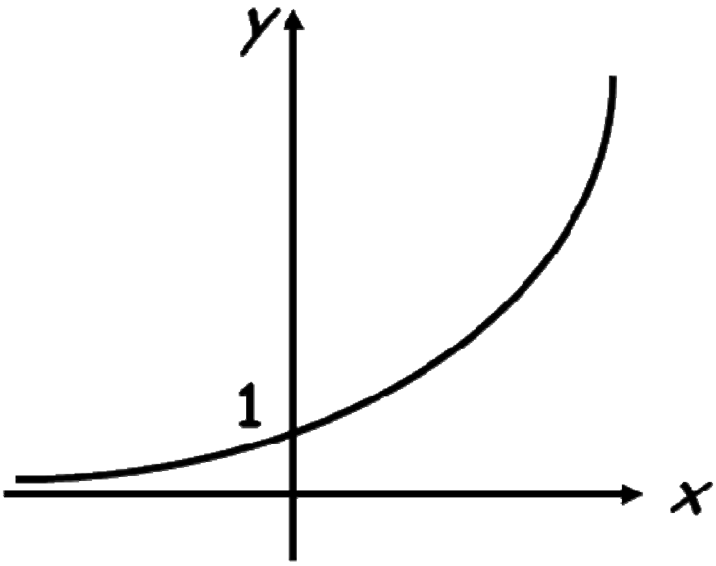
Ex1

Draw the graph with equation $y = 2^x$.

Start with a table of values:

x	-3	-2	-1	0	1	2	3
y	2^{-3}	2^{-2}	2^{-1}	2^0	2^1	2^2	2^3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

If you were to plot this the shape you would see is:

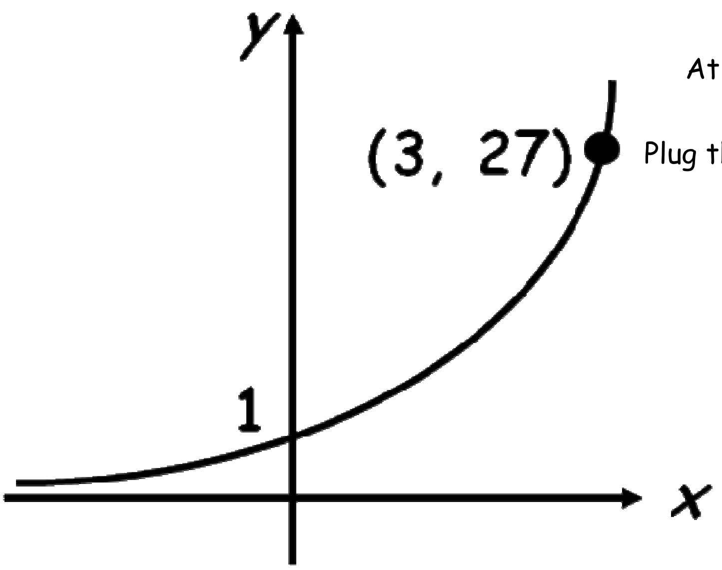


This shape is called an exponential.

The *x* axis is called an asymptote to the curve since the curve gets closer and closer to it but never meets it.

Ex2

This curve is of the form $y = a^x$. If one of the coordinates on this curve is (3, 27), what is the constant *a*?



At (3, 27), *x* = 3 and *y* = 27.

Plug this into the equation $y = a^x$

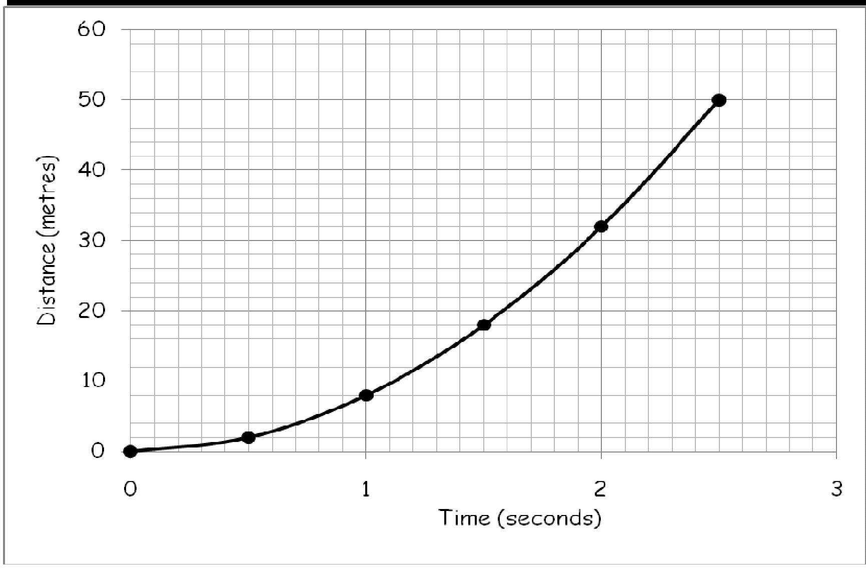
$$y = a^x$$
$$27 = a^3$$
$$a = \sqrt[3]{27}$$
$$a = 3$$

Mathematical Models

Ex1

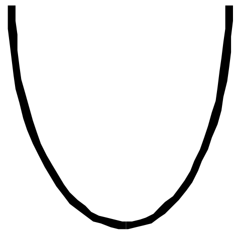
Fast Car Research is testing a new racing car. The table shows the data collected and a graph of the data has been produced.

Time (x seconds)	0	0.5	1.0	1.5	2.0	2.5
Distance (y metres)	0	2	8	18	32	50



Is it possible to get an equation for this curve?

What curve does it look like?



Quadratic??

$y = ax^2$??

We can find the equation of this curve by substituting in an x and
Choose the point (1, 8)

$y = ax^2$

$8 = a \times 1^2$

$\therefore a = 8$

This gives the equation: $y = 8x^2$

We can now *use* the equation for other values that are not plotted.

(a) What distance has the car travelled when the car been going for 2.2secs?

$$y = 8x^2$$
$$y = 8 \times 2.2^2$$
$$y = 38.72m$$

(b) At what time had the car gone 200m?

$$
\begin{aligned}
y &= 8x^2 \\
200 &= 8x^2 \\
x^2 &= \frac{200}{8} \\
x^2 &= 25 \\
x &= \sqrt{25} \\
x &= 5secs
\end{aligned}
$$