MATHEMATICS
NATIONAL 5 NOTES

$\times$

$\pm$

$\div$

$A = \pi r^2$

$\leq$

$\sum$

x

π

$V = lbh$

$\sqrt{}$

$+$

$-$

INDEX:

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FORMULAE LIST

The roots of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine rule: $a^2 = b^2 + c^2 - 2bc \cos A$ or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Area of a triangle: $\text{Area} = \frac{1}{2} ab \sin C$

Volume of a sphere: $\text{Volume} = \frac{4}{3}\pi r^3$

Volume of a cone: $\text{Volume} = \frac{1}{3}\pi r^2 h$

Volume of a pyramid: $\text{Volume} = \frac{1}{3} Ah$

Volume of a cylinder: $\text{Volume} = \pi r^2 h$

Standard deviation: $s = \sqrt{\dfrac{\sum(x - \bar{x})^2}{n-1}} = \sqrt{\dfrac{\sum x^2 - (\sum x)^2 / n}{n-1}}$, where n is the sample size.

CHAPTER 1: FRACTIONS

Simplify (cancel down):

$$\frac{28}{36} = \frac{28 \div 4}{36 \div 4} = \frac{7}{9}$$

*divide by **highest common factor**, HCF(28,36) = 4*

Mixed numbers and **improper fractions** (top heavy):

$$\overrightarrow{4 \times \mathbf{2} + 3 = 11}$$

$$2\frac{3}{4} = \mathbf{2} + \frac{3}{4} = \frac{\mathbf{8}}{\mathbf{4}} + \frac{3}{4} = \frac{11}{4}$$

'bottom' stays as 4

$$\overleftarrow{11 \div 4 = \mathbf{2} \ R3}$$

Addition and Subtraction requires a common denominator:

$$\frac{5}{6} - \frac{4}{9}$$

least common multiple LCM(6,9)=18

$$= \frac{15}{18} - \frac{8}{18}$$

$$\frac{5 \times 3}{6 \times 3} = \frac{15}{18} \qquad \frac{4 \times 2}{9 \times 2} = \frac{8}{18}$$

both fractions now 18ths

$$= \frac{7}{18}$$

subtract 'top' numbers, still 18ths

Multiplication and Division requires **no** mixed numbers:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \qquad\qquad \frac{a}{b} \div \frac{\mathbf{c}}{\mathbf{d}} = \frac{a}{b} \times \frac{\mathbf{d}}{\mathbf{c}}$$

$$\frac{3}{10} \times 2\frac{3}{4} \qquad\qquad \frac{4}{5} \div \frac{7}{8}$$

$$= \frac{3}{10} \times \frac{11}{4} \quad \textit{go 'top heavy'} \qquad = \frac{4}{5} \times \frac{8}{7} \quad \textit{multiply by reciprocal}$$

$$= \frac{33}{40} \qquad\qquad = \frac{32}{35}$$

ORDER OF CALCULATION: $\times$ *and* $\div$ *before* $+$ *or* $-$

to change the order use brackets *BRACKETS FIRST!*

Carry out separate calculations:

(1) $\quad 7\dfrac{5}{6} \ - \ \dfrac{8}{9} \ of \ 2\dfrac{3}{4}$

multiply first

$\dfrac{8}{9} \ \times \ \dfrac{11}{4}$ **of** *means* **multiply**

$= \dfrac{88}{36}$ *fully simplify*

$= \dfrac{22}{9}$

$= 2\dfrac{4}{9}$

subtraction last

$7\dfrac{5}{6} \ - \ 2\dfrac{4}{9}$

$= 7\dfrac{15}{18} \ - \ 2\dfrac{8}{18}$ *common denominator*

$= 5\dfrac{7}{18}$ *subtract whole numbers and fractions separately*

(2) $\quad \left(\dfrac{5}{6} \ - \ \dfrac{1}{2}\right) \ \div \ 1\dfrac{3}{4}$

$\dfrac{5}{6} \ - \ \dfrac{1}{2}$ *brackets first*

$= \dfrac{5}{6} \ - \ \dfrac{3}{6}$

$= \dfrac{2}{6}$ *fully simplify*

$= \dfrac{1}{3}$

$\dfrac{1}{3} \ \div \ 1\dfrac{3}{4}$

$= \dfrac{1}{3} \ \div \ \dfrac{7}{4}$ *'top-heavy' first*

$= \dfrac{1}{3} \ \times \ \dfrac{4}{7}$ *multiply by reciprocal*

$= \dfrac{4}{21}$

CHAPTER 2: PERCENTAGES

PERCENTAGE CHANGE

original value → changed value

INCREASE: growth, appreciation, compound interest $\quad 100\% \xrightarrow{+a\%} (100 + a)\%$

DECREASE: decay, depreciation $\qquad\qquad\qquad 100\% \xrightarrow{-a\%} (100 - a)\%$

For example,

8% increase: $100\% \xrightarrow{+8\%} 108\% = 1\cdot08 \quad$ multiply quantity by $1\cdot08$ for 8% increase

8% decrease: $100\% \xrightarrow{-8\%} 92\% = 0\cdot92 \quad$ multiply quantity by $0\cdot92$ for 8% decrease

REVERSING PERCENTAGE CHANGE

Divide by the factor which produced the increase.

(1) Including VAT of 20%, a radio costs £96. Find the original cost exclusive of VAT.

20% VAT added

$100\% \xrightarrow{+20\%} 120\% = 1\cdot20$

$£x \times 1\cdot20 = £96$
$£x = £96 \div 1\cdot20$
$\quad\;\; = £80$

non-calculator: $\quad 120\% \xrightarrow{\div12} 10\% \xrightarrow{\times10} 100\%$

$£96 \div 12 = £8 \qquad £8 \times 10 = £80$

(2) A camera costs £120 after a discount of 25% is applied. Find the original cost.

25% discount subtracted

$100\% \xrightarrow{-25\%} 075\% = 0\cdot75$

$£x \times 0\cdot75 = £120$
$£x = £120 \div 0\cdot75$
$\quad\;\; = £160$

non-calculator: $\quad 75\% = \dfrac{3}{4} \qquad \dfrac{3}{4} \xrightarrow{\div3} \dfrac{1}{4} \xrightarrow{\times4} \dfrac{4}{4}$

$£120 \div 3 = £40 \qquad £40 \times 4 = £160$

COMPOUND PERCENTAGE CHANGE

appreciation and depreciation

(1) A £240 000 house appreciates in value by 5% in 2007, appreciates 10% in 2008 and depreciates by 15% in 2009. Calculate the value of the house at the end of 2009.

or   *evaluate year by year*

year 1

5% *increase*: $100\% + 5\% = 105\% = 1 \cdot 05$ 5% *of* £240000 = £12000

10% *increase*: $100\% + 10\% = 110\% = 1 \cdot 10$ £240000 + £12000 = £25200

15% *decrease*: $100\% - 15\% = 085\% = 0 \cdot 85$

year 2

10% *of* £252000 = £25200

£252000 + £25200 = £277200

year 3

£240000 $\times 1 \cdot 05 \times 1 \cdot 10 \times 0 \cdot 85$ 15% *of* £277200 = £41580

= £235620   £277200 − £41580 = £235620

compound interest

(2) Calculate the compound interest on £12000 invested at 5% pa for 3 years.

£12000 $\times (1 \cdot 05)^3$ *ie.* $\times 1 \cdot 05 \times 1 \cdot 05 \times 1 \cdot 05$ *or evaluate year by year*

£12000 $\times 1 \cdot 157625$

= £13891 $\cdot$ 50

compound interest = £13891 $\cdot$ 50 − £12000 = £1891 $\cdot$ 50

EXPRESSING CHANGE AS A PERCENTAGE

$$\% \ change = \frac{change}{start} \times 100\%$$

A £15000 car is resold for £12000.   *loss* = £15000 − £12000 = £3000
Find the percentage loss.

$$\% \ loss = \frac{3000}{15000} \times 100\% = 20\%$$

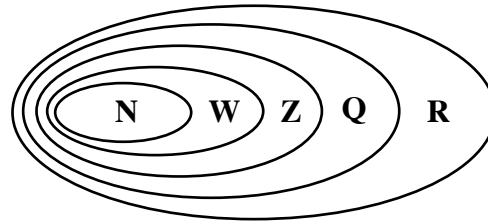# CHAPTER 3: SURDS

NUMBER SETS:

Natural numbers   N =   {1, 2, 3...}
Whole numbers   W = {0, 1, 2, 3...}
Integers Z = {... -3, -2, -1, 0, 1, 2, 3...}

Rational numbers, Q , can be written as a division of two integers.
Irrational numbers **cannot** be written as a division of two integers.

Real numbers, R , are all rational and irrational numbers.

SURDS ARE IRRATIONAL ROOTS.

For example, $\sqrt{2}$, $\sqrt{\dfrac{5}{9}}$, $\sqrt[3]{16}$ are surds.

whereas $\sqrt{25}$, $\sqrt{\dfrac{4}{9}}$, $\sqrt[3]{-8}$ are **not** surds as they are 5 , $\dfrac{2}{3}$ and -2 respectively.

SIMPLIFYING ROOTS:

RULES: $\sqrt{mn} = \sqrt{m} \times \sqrt{n}$ $\sqrt{\dfrac{m}{n}} = \dfrac{\sqrt{m}}{\sqrt{n}}$

(1) Simplify $\sqrt{24} \times \sqrt{3}$

$\sqrt{24} \times \sqrt{3}$
$= \sqrt{72}$

$= \sqrt{36} \times \sqrt{2}$ *36 is the largest square number which is a factor of 72*

$= 6 \times \sqrt{2}$
$= 6\sqrt{2}$

(2) Simplify $\sqrt{72} + \sqrt{48} - \sqrt{50}$

$\sqrt{72}$ + $\sqrt{48}$ – $\sqrt{50}$

$= \sqrt{36} \times \sqrt{2}$ + $\sqrt{16} \times \sqrt{3}$ – $\sqrt{25} \times \sqrt{2}$

$=$ $6\sqrt{2}$ + $4\sqrt{3}$ – $5\sqrt{2}$

$=$ $6\sqrt{2} - 5\sqrt{2}$ + $4\sqrt{3}$

$=$ $\sqrt{2}$ + $4\sqrt{3}$

(3) Remove the brackets and fully simplify:

(a) $\left(\sqrt{3} - \sqrt{2}\right)^2$

$= \left(\sqrt{3} - \sqrt{2}\right)\left(\sqrt{3} - \sqrt{2}\right)$

$= \sqrt{3}\left(\sqrt{3} - \sqrt{2}\right) - \sqrt{2}\left(\sqrt{3} - \sqrt{2}\right)$

$= \quad \sqrt{9} - \sqrt{6} \quad - \quad \sqrt{6} + \sqrt{4}$

$= \quad 3 - \sqrt{6} \quad - \quad \sqrt{6} + 2$

$= \quad 5 - 2\sqrt{6}$

(b) $\left(3\sqrt{2} + 2\right)\left(3\sqrt{2} - 2\right)$

$= \left(3\sqrt{2} + 2\right)\left(3\sqrt{2} - 2\right)$

$= 3\sqrt{2}\left(3\sqrt{2} - 2\right) + 2\left(3\sqrt{2} - 2\right)$

$= \quad 9\sqrt{4} - 6\sqrt{2} \quad + 6\sqrt{2} \quad - 4$

$= \quad 18 - 6\sqrt{2} \quad + 6\sqrt{2} \quad - 4$

$= \quad 14$

RATIONALISING DENOMINATORS:
Removing surds from the denominator.

Express with a rational denominator:

(1) $\dfrac{4}{\sqrt{6}}$

$\dfrac{4}{\sqrt{6}}$

$= \dfrac{4 \times \sqrt{6}}{\sqrt{6} \times \sqrt{6}}$ *multiply the 'top' and 'bottom'*
               by the surd on the denominator

$= \dfrac{4\sqrt{6}}{6}$

$= \dfrac{2\sqrt{6}}{3}$

(2) $\dfrac{\sqrt{3}}{3\sqrt{2}}$

$\dfrac{\sqrt{3}}{3\sqrt{2}}$

$= \dfrac{\sqrt{3} \times \sqrt{2}}{3\sqrt{2} \times \sqrt{2}}$

$= \dfrac{\sqrt{6}}{3 \times \sqrt{4}}$

$= \dfrac{\sqrt{6}}{6}$

CHAPTER 4: INDICES

$$\text{base} \longrightarrow a^n \longleftarrow \text{index or exponent}$$

INDICES RULES: require the same base.   **Examples:**

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$\frac{w^2 \times w^5}{w^3} = \frac{w^7}{w^3} = w^4$$

$$\left(a^m\right)^n = a^{mn}$$

$$\left(3^5\right)^2 = 3^{10}$$

$$\left(ab\right)^n = a^n b^n$$

$$\left(2a^3 b\right)^2 = 2^2 a^6 b^2 = 4a^6 b^2$$

$$a^0 = 1$$
$$a^1 = a$$

$$5^0 = 1$$
$$5^1 = 5$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

$$8^{\frac{4}{3}} = \left(\sqrt[3]{8}\right)^4 = 2^4 = 16$$

$$\frac{1}{a^p} = a^{-p}$$

$$8^{-\frac{4}{3}} = \frac{1}{8^{\frac{4}{3}}} = \frac{1}{16}$$

SCIENTIFIC NOTATION (STANDARD FORM)

Writing numbers in the form $a \times 10^n$

where $1 \leq a < 10$ *place the decimal point after the first non-zero digit*
and *n* is an integer   *numbers ...-3, -2, -1, 0, 1, 2, 3....*

32800 $= 3 \cdot 28 \times 10 \times 10 \times 10 \times 10 = \mathbf{3 \cdot 28 \times 10^4}$

0·000328 $= 3 \cdot 28 \div 10 \div 10 \div 10 \div 10 = 3 \cdot 28 \div 10^4 = \mathbf{3 \cdot 28 \times 10^{-4}}$

Notice for numbers starting $0\cdot$ the power of 10 is negative (same as $\div 10$).

SIGNIFICANT FIGURES indicate the accuracy of a **measurement**.
For example,   3400 cm = 34 m = 0·034 km
                same measurement, same accuracy, each 2 significant figures.

Count the number of figures used, but **not** zeros at the **end** of a whole number
                or zeros at the **start** of a decimal.

rounding: 5713·4   5700 to 2 significant figures
        0·057134   0·057 to 2 significant figures   (note: 0·057000 wrong)

(1)   One **milligram** of hydrogen gas contans $2 \cdot 987 \times 10^{20}$ molecules.
    Calculate, to 3 significant figures, the number of molecules in **5 grams** of hydrogen.

$5000 \times 2 \cdot 987 \times 10^{20}$ *learn to enter standard form in the calculator using the*
$= 1 \cdot 4935 \times 10^{24}$ *appropriate button* EE *or* EXP *or* x10^n *eg.* 2·987 EXP *20*
$\approx 1 \cdot 49 \times 10^{24}$ molecules

(2)   The total mass of argon in a flask is $4 \cdot 15 \times 10^{-2}$ grams.
    The mass of a single atom of argon is $6 \cdot 63 \times 10^{-23}$ grams.

    Find, correct to 3 significant figures, the number of argon atoms in the flask.
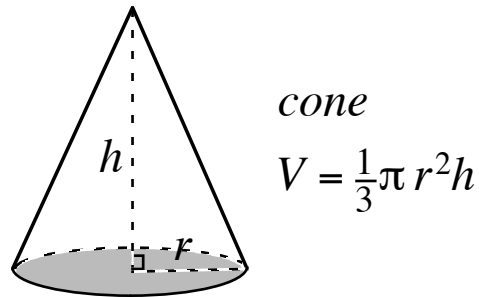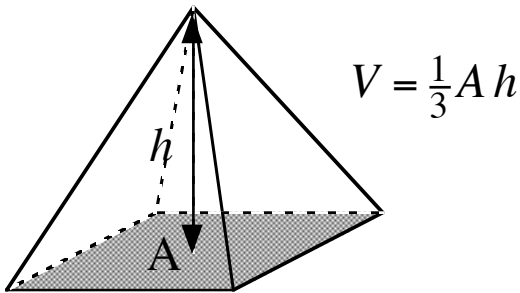
$\dfrac{4 \cdot 15 \times 10^{-2}}{6 \cdot 63 \times 10^{-23}}$ *use the (–) button for a minus   eg.* 4·15 EXP (–) 2

$= 6 \cdot 259... \times 10^{20}$ *divide 'top' by 'bottom' and write the unrounded answer*

$\approx 6 \cdot 26 \times 10^{20}$ atoms   *write the rounded answer*

CHAPTER 5: VOLUMES OF SOLIDS

PRISMS   a solid with the same cross-section throughout its length.
length l is at right-angles to the area A.

$$V = A\,l$$

cylinder $V = \pi r^2 h$

PYRAMIDS

$$V = \tfrac{1}{3} A\,h$$

cone

$$V = \tfrac{1}{3}\pi\, r^2 h$$

SPHERE

$$V = \tfrac{4}{3}\pi\, r^3$$
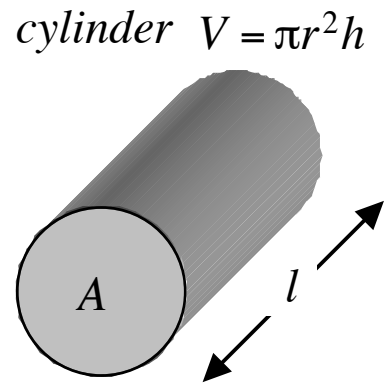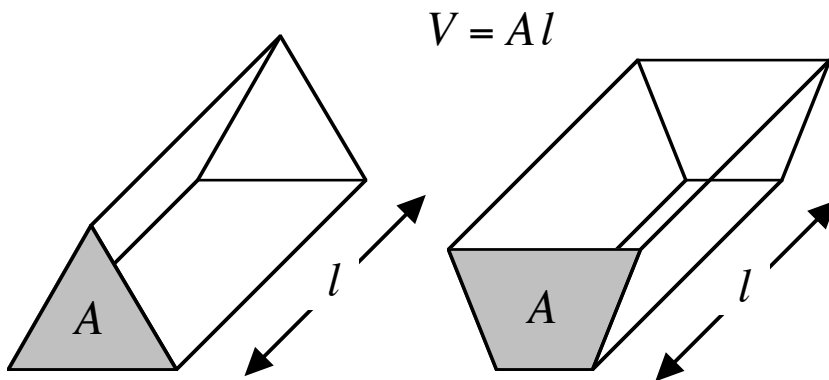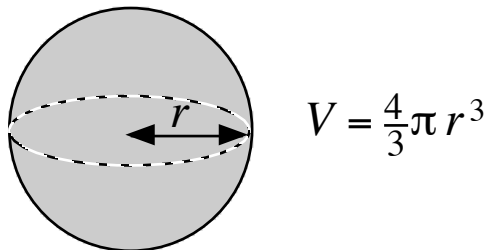
EFFECT OF CHANGE

Describe the effect on the volume of a cylinder of:

(i) trebling the radius   (ii) doubling the radius and halving the height

$$V = \pi(3r)^2 h$$

$$= \pi 9 r^2 h$$

$$= 9\pi r^2 h$$

     9 *times bigger*

$$V = \pi(2r)^2\left(\tfrac{1}{2}h\right)$$

$$= \pi 4 r^2 \times \tfrac{1}{2}h$$

$$= 2\pi r^2 h$$

     2 *times bigger*

USING THE FORMULAE

(1) Calculate the volume.

$$A = \tfrac{1}{2}bh$$
$$= 10 \times 8 \div 2$$
$$= 40 \ cm^2$$

$$V = Al$$
$$= 40 \times 20$$
$$= 800 \ cm^3$$

8 cm

20 cm

8 cm
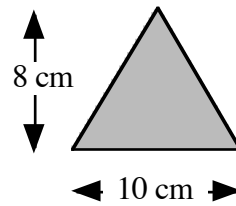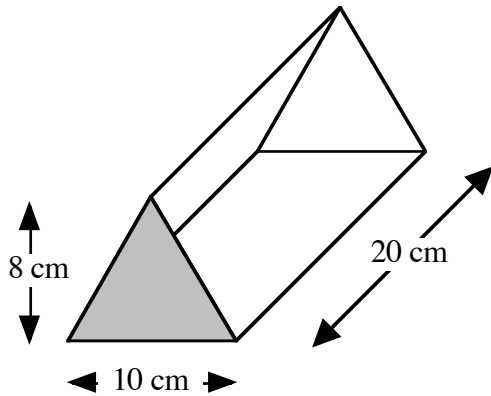
10 cm

10 cm

(2) Calculate the volume correct to **3 significant figures**.

25 cm

30 cm
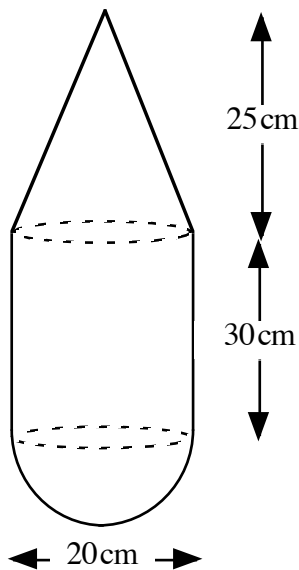
20 cm

$$radius = 20cm \div 2 = 10cm$$

$$V = \tfrac{1}{3}\pi r^2 h$$
$$= \tfrac{1}{3} \times \pi \times 10 \times 10 \times 25$$
$$= 2617 \cdot 993... \ cm^3$$

$$V = \pi r^2 h$$
$$= \pi \times 10 \times 10 \times 30$$
$$= 9424 \cdot 777... \ cm^3$$

$$V = \tfrac{4}{3}\pi r^3 \div 2$$
$$= \tfrac{4}{3} \times \pi \times 10 \times 10 \times 10 \div 2$$
$$= 2094 \cdot 395... \ cm^3$$

$$total \ volume = 2617 \cdot 993... + 9424 \cdot 777... + 2094 \cdot 395...$$
$$= 14137 \cdot 166...$$
$$\approx 14100 \ cm^3$$

CHAPTER 6: PYTHAGORAS' THEOREM
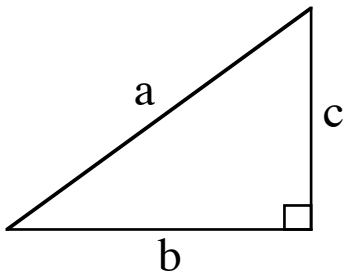
Side length **a** units is opposite the right angle. It is the longest side, the **hypotenuse**.

For **right-angled triangles**:

$$a^2 = b^2 + c^2$$

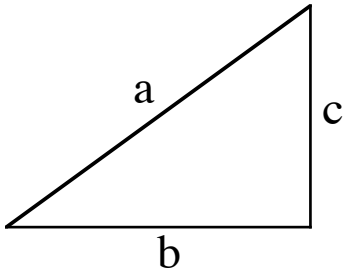**3D** Identify right-angled triangles in 3D diagrams.
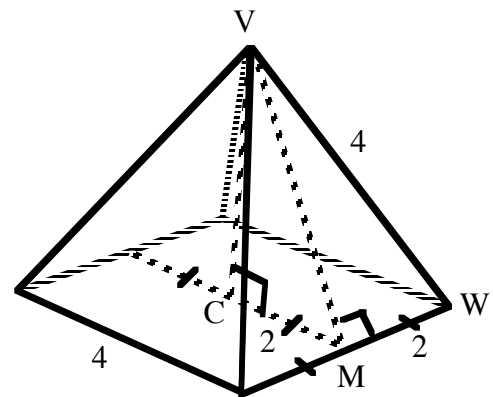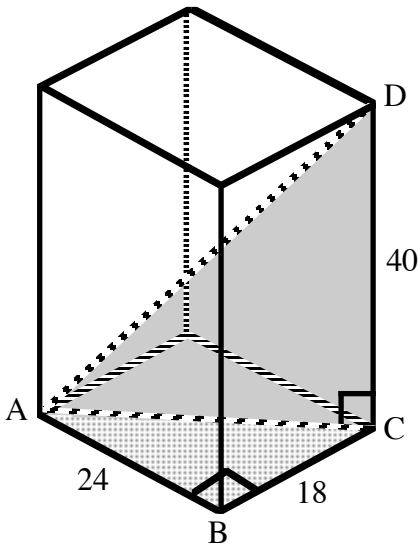
face diagonal
$$AC^2 = 24^2 + 18^2$$
$$= 576 + 324$$
$$= 900$$
$$AC = \sqrt{900}$$
$$= 30$$

slant height
$$VM^2 = 4^2 - 2^2$$
$$= 16 - 4$$
$$= 12$$
$$VM = \sqrt{12}$$
$$= 2\sqrt{3}$$

space diagonal
$$AD^2 = 40^2 + 30^2$$
$$= 1600 + 900$$
$$= 2500$$
$$AD = \sqrt{2500}$$
$$= 50$$

height
$$VC^2 = \left(\sqrt{12}\right)^2 - 2^2$$
$$= 12 - 4$$
$$= 8$$
$$VC = \sqrt{8}$$
$$= 2\sqrt{2}$$

CONVERSE OF PYTHAGORAS' THEOREM

Longest side, length **a** units, is opposite the biggest angle.

If $\qquad\qquad a^2 = b^2 + c^2$

then $\qquad\qquad$ the triangle is right-angled

Show that ΔABC is right angled.
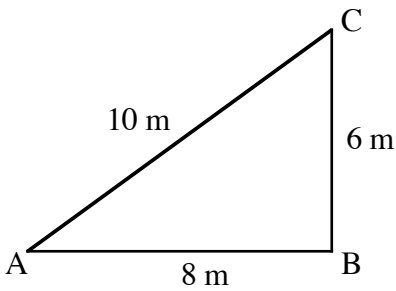
$AB^2 + BC^2 = 8^2 + 6^2 = 100$
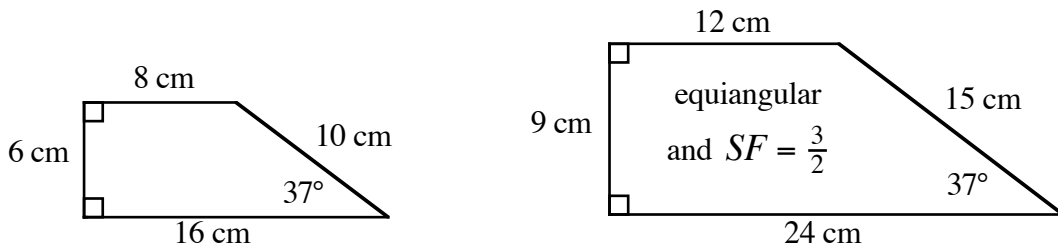
$AC^2 = 10^2 = 100$

since $AB^2 + BC^2 = AC^2$

by the Converse of Pyth. Thm.

ΔABC *is right - angled at B* (*ie.* $\angle ABC = 90°$)

CHAPTER 7: SIMILAR SHAPES

Shapes are similar if they are enlargement or reductions of each other.

(1) the angles remain unchanged - the shapes are **equiangular**.

and (2) the sides are enlarged or reduced by some **scale factor** (SF).

8 cm

6 cm

10 cm

37°

16 cm

12 cm

equiangular

and $SF = \frac{3}{2}$

9 cm

15 cm

37°

24 cm

Triangles are special:

Enlarge or reduce sides by some scale factor and the two triangles will be equiangular.

If triangles are equiangular then they are similar.

SCALING LENGTH

length scale factor, $SF = \dfrac{\text{image side}}{\text{original side}}$

enlargement if $SF > 1$

reduction if $0 < SF < 1$

16 cm x cm

6 cm

9 cm

Find the value of x.

o   Original

16 cm

x

15 cm

Δs equiangular and so similar

o   Image

x cm

x   6 cm

shape with the unknown side
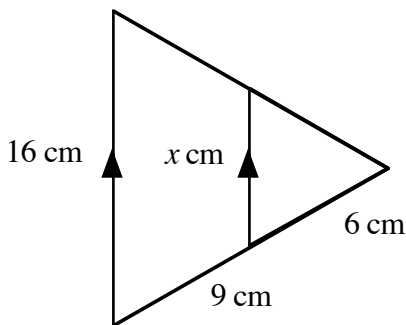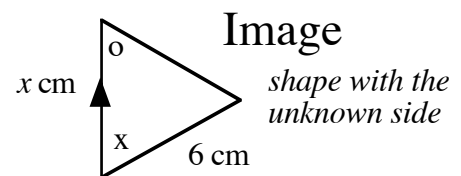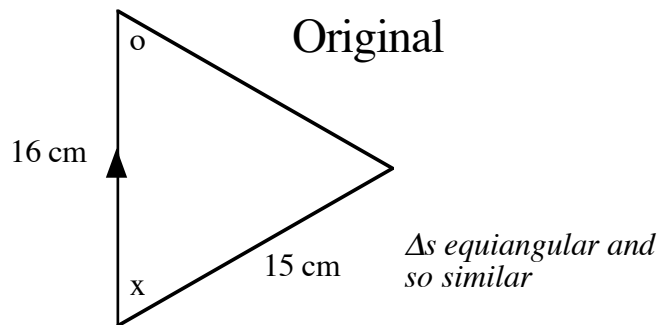
$SF = \dfrac{image}{original} = \dfrac{6}{15} = \dfrac{2}{5}$ *0 < SF < 1 as expected for a reduction*
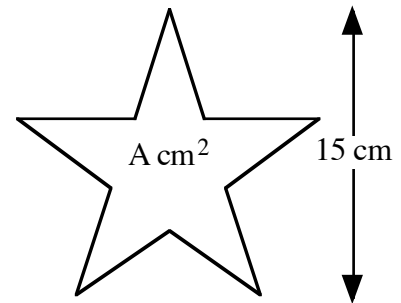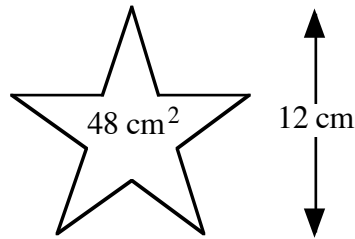
$x = \dfrac{2}{5} \times 16 = 6 \cdot 4$ *smaller than 16 as expected for a reduction*

SCALING AREA   for a 2D shape both length and breadth must be scaled.

length SF = n
area SF = n^2

Given that the two shapes shown are similar, find the area of the larger shape.

$$length \text{ SF} = \frac{image}{original} = \frac{15}{12} = \frac{5}{4}$$ *SF > 1 as expected for an enlargement*

$$area \text{ SF} = \frac{5}{4} \times \frac{5}{4} = \frac{25}{16}$$
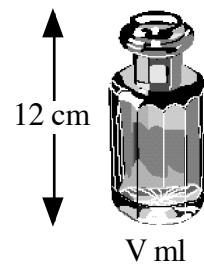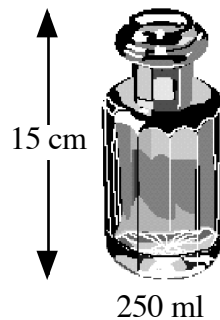
$$A = \frac{25}{16} \times 48 = 75$$ *bigger than 48 as expected for an enlargement*

SCALING VOLUME   for a 3D shape length, breadth and height must be scaled.

length SF = n
volume SF = n^3

Given that the two solids shown are similar find the volume of the smaller solid.

$$length \text{ SF} = \frac{image}{original} = \frac{12}{15} = \frac{4}{5}$$ *0 < SF < 1 as expected for a reduction*

$$volume \text{ SF} = \frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} = \frac{64}{125}$$
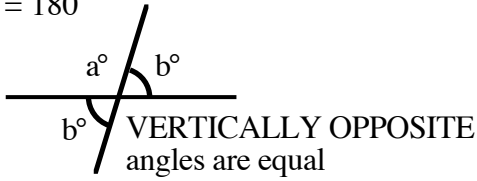
$$V = \frac{64}{125} \times 250 = 128$$ *smaller than 250 as expected for a reduction*
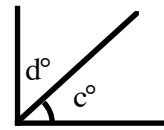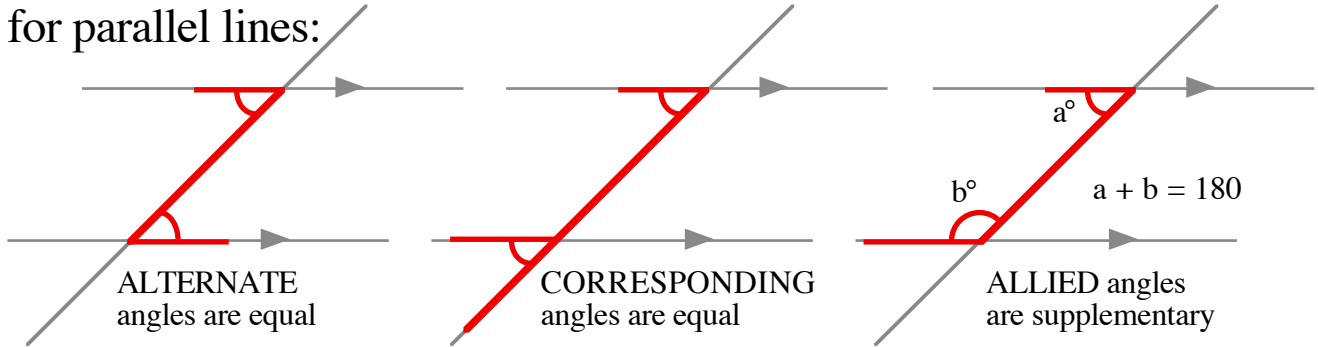
CHAPTER 8: ANGLES and SHAPE

LINES

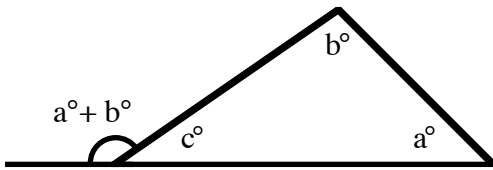SUPPLEMENTARY angles
a + b = 180

COMPLEMENTARY angles
c + d = 90

a° b°

d°
c°

b° VERTICALLY OPPOSITE
angles are equal

for parallel lines:

ALTERNATE
angles are equal

CORRESPONDING
angles are equal

a°

b°

a + b = 180

ALLIED angles
are supplementary

TRIANGLES

angle sum   a°+ b°+ c° = 180°

a°+ b°

b°

c°

a°

EXTERIOR ANGLE is the sum of the
opposite INTERIOR ANGLES

ISOSCELES

60°

60°   60°
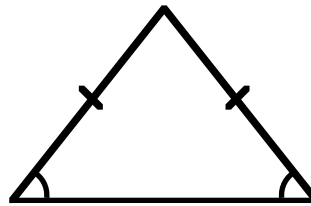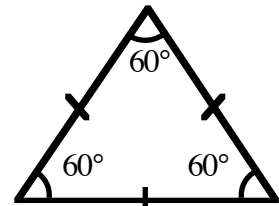
EQUILATERAL

POLYGONS

INTERIOR ANGLES   sum to (n-2)×180°   where n sides
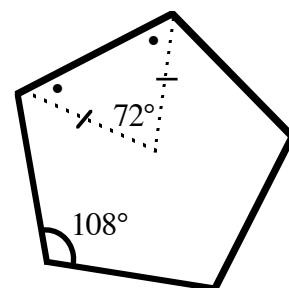EXTERIOR ANGLES   sum to 360°

all sides and angles equal

72°

108°
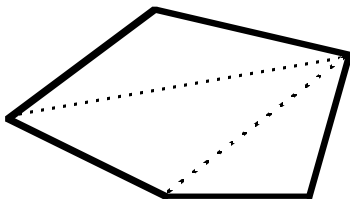
pentagon
interior sum 3×180° = 540°

exterior sum 360°

REGULAR pentagon
interior angle 540°÷5 = 108°

QUADRILATERALS

KITE

TRAPEZIUM

$a + b = 180$, $c + d = 180$
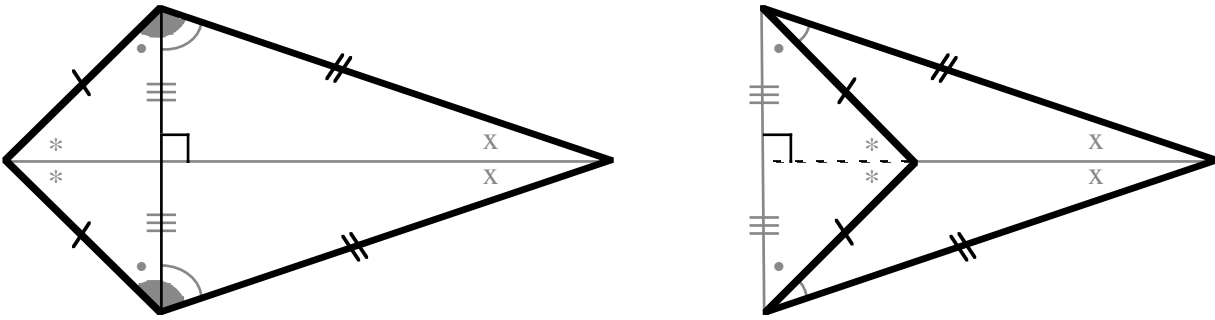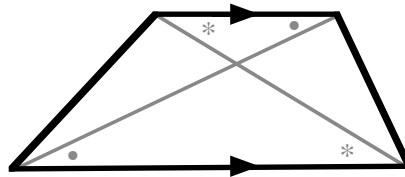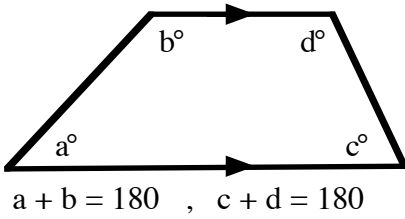
PARALLELOGRAM a trapezium
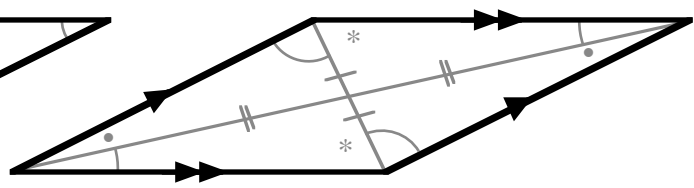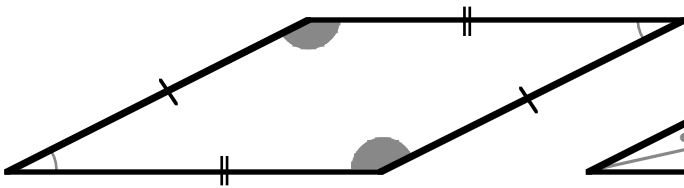
RHOMBUS a kite and parallelogram
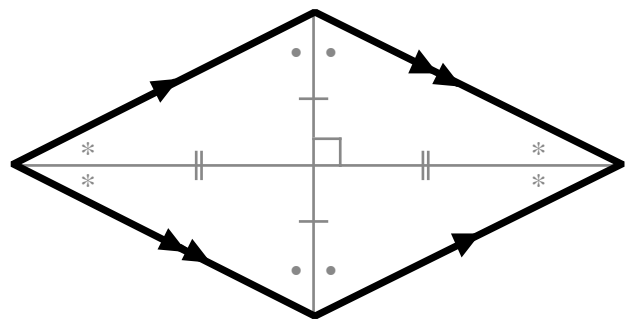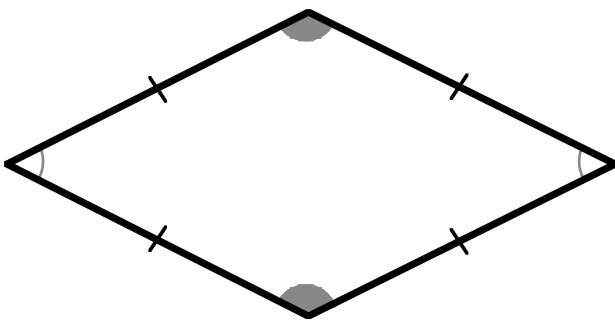
RECTANGLE a parallelogram
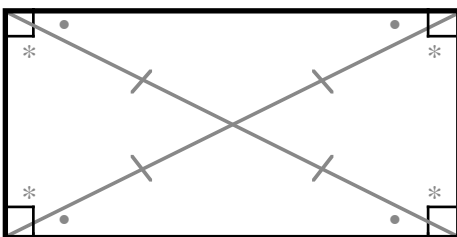
SQUARE a rectangle and rhombus.

page 16
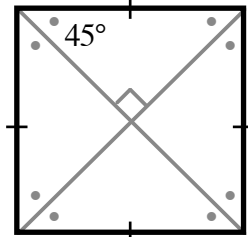
CHAPTER 9: PROPERTIES OF THE CIRCLE

angle in a semicircle is a right-angle.	the perpendicular bisector of a chord is a diameter.	a tangent and the radius drawn to the point of contact form a right-angle.
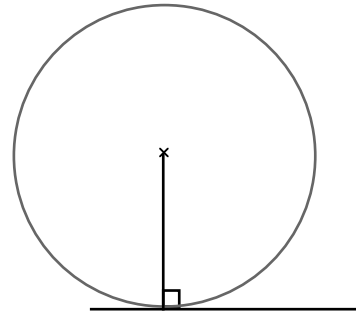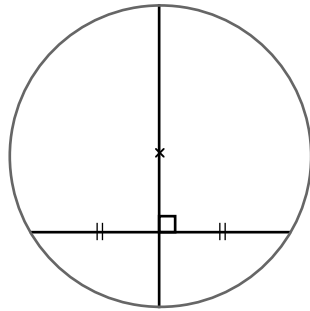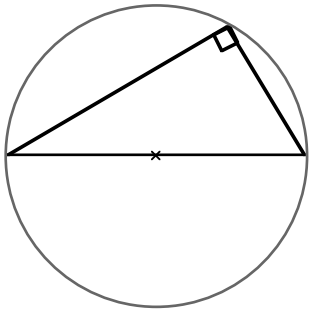
ANGLES

(1)
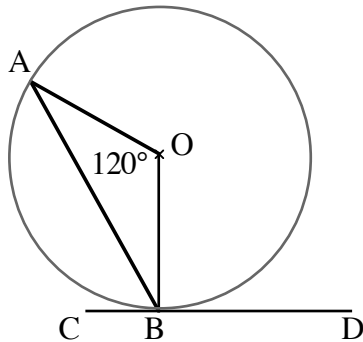
radius OA = OB so ΔAOB is isosceles and Δ angle sum 180° :

$$\angle OBA = (180° - 120°) \div 2 = 30°$$

tangent CD and radius OB : $\angle OBC = 90°$

Calculate the size of angle ABC.　　$\angle ABC = 90° - 30° = 60°$

(2)
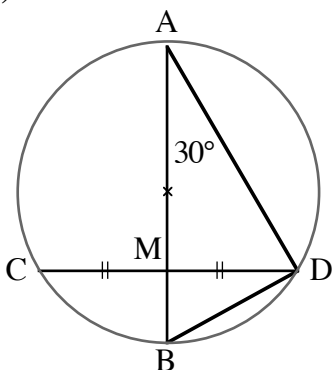
diameter AB bisects chord CD : $\angle AMD = 90°$

ΔAMD angle sum 180° :
$$\angle ADM = 180° - 90° - 30° = 60°$$

angle in a semicircle : $\angle ADB = 90°$

Calculate the size of angle BDC.　　$\angle BDC = 90° - 60° = 30°$

PYTHAGORAS' THEOREM
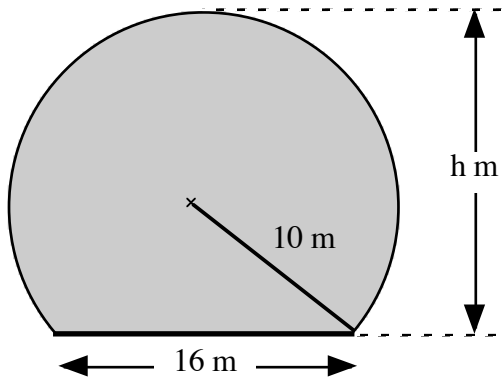
A circular road tunnel, radius 10 metres, is cut through a hill.

The road has a width 16 metres.

Find the height of the tunnel.
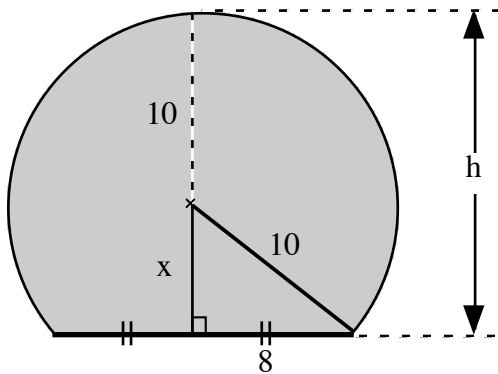
the diameter drawn is the perpendicular bisector of the chord:
Δ *is right-angled so can apply Pyth. Thm.*

$$x^2 = 10^2 - 8^2$$
$$= 100 - 64$$
$$= 36$$
$$x = \sqrt{36}$$
$$x = 6$$

$$h = x + 10$$
$$= 6 + 10$$
$$h = 16$$

height 16 metres

CHAPTER 10: ARCS and SECTORS
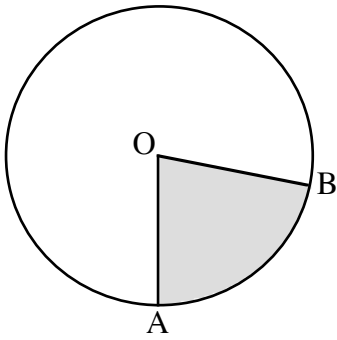
$$\frac{\angle AOB}{360°} = \frac{arc\ AB}{\pi d} = \frac{area\ of\ sector\ AOB}{\pi r^2}$$

Choose the appropriate pair of ratios based on:

(i) the ratio which includes the quantity to be found

(ii) the ratio for which both quantities are known

(or can be found).

(1) Find the **exact** length of **major** arc AB.
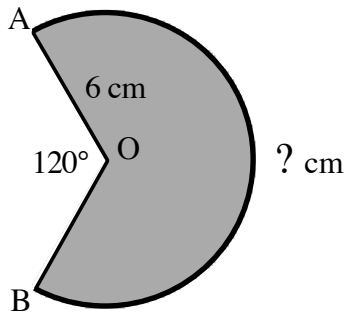
$$\frac{\angle AOB}{360°} = \frac{arc\ AB}{\pi d}$$

$$\frac{240°}{360°} = \frac{arc\ AB}{\pi \times 12}$$

$$arc\ AB = \frac{240°}{360°} \times \pi \times 12$$

$$= 8\pi\ cm \quad (25 \cdot 132...)$$

diameter $d = 2 \times 6\ cm = 12\ cm$

reflex $\angle AOB = 360° - 120° = 240°$

(2) Find the **exact** area of sector AOB.
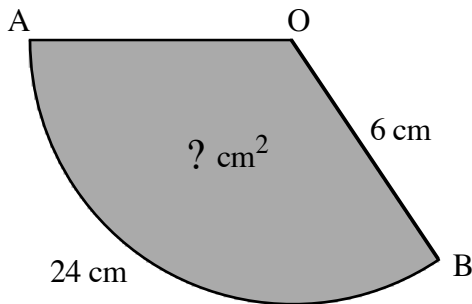
$$\frac{arc\ AB}{\pi d} = \frac{area\ of\ sector\ AOB}{\pi r^2}$$

$$\frac{24}{\pi \times 12} = \frac{area\ of\ sector\ AOB}{\pi \times 6 \times 6}$$

$$area\ of\ sector\ AOB = \frac{24}{\pi \times 12} \times \pi \times 6 \times 6$$

$$= 72\ cm^2$$

CHAPTER 11: ALGEBRAIC EXPRESSIONS
REMOVING BRACKETS

SINGLE BRACKETS

(1) $3x(2x - y + 7)$

$= 6x^2 - 3xy + 21x$

$3x \times 2x = 6x^2$
$3x \times -y = -3xy$
$3x \times +7 = +21x$

(2) $-2(3t + 5)$

$= -6t - 10$

$-2 \times 3t = -6t$
$-2 \times +5 = -10$

(3) $-3w(w^2 - 4)$

$= -3w^3 + 12w$

$-3w \times w^2 = -3w^3$
$-3w \times -4 = +12w$

Fully simplify:

(4) $2t(3 - t) + 5t^2$

$= 6t - 2t^2 + 5t^2$
$= 6t + 3t^2$

(5) $5 - 3(n - 2)$

$= 5 - 3n + 6$
$= 5 + 6 - 3n$
$= 11 - 3n$

DOUBLE BRACKETS

(1) $(3x + 2)(2x - 5)$

$= 3x(2x - 5) + 2(2x - 5)$

$= 6x^2 - 15x + 4x - 10$

$= 6x^2 - 11x - 10$

or

"FOIL"

$(3x + 2)(2x - 5)$

(2) $(2t - 3)^2$

$= (2t - 3)(2t - 3)$

$= 2t(2t - 3) - 3(2t - 3)$

$= 4t^2 - 6t - 6t + 9$

$= 4t^2 - 12t + 9$

(3) $(w + 2)(w^2 - 3w + 5)$

$= w(w^2 - 3w + 5) + 2(w^2 - 3w + 5)$

$= w^3 - 3w^2 + 5w + 2w^2 - 6w + 10$

$= w^3 - 3w^2 + 2w^2 + 5w - 6w + 10$

$= w^3 - w^2 - w + 10$

FACTORSATION

COMMON FACTORS $\qquad ab + ac = a(b + c)$

Highest Common Factors are used to write expressions in **fully** factorised form.

Factorise **fully**: $\qquad 4a - 2a^2$

$$= 2a(2 - a) \qquad\qquad 2a \times 2 - 2a \times a \ \ using \ HCF(4a, 2a^2) = 2a$$

NOTE: the following answers are factorised but not **fully** factorised:

$$2(2a - a^2)$$
$$a(4 - 2a)$$

DIFFERENCE OF TWO SQUARES $\qquad a^2 - b^2 = (a + b)(a - b)$

Factorise **fully**:

(1) $4x^2 - 9$ $\qquad\qquad\qquad$ (2) $4x^2 - 36$

$= (2x)^2 - 3^2$ $\qquad\qquad\qquad = 4(x^2 - 9) \qquad\qquad$ *common factor first*

$= (2x + 3)(2x - 3)$ $\qquad\qquad\quad = 4(x + 3)(x - 3)$

$\qquad\qquad\qquad\qquad$ NOTE: $(2x + 6)(2x - 6)$ is factorised but not **fully** factorised.

TRINOMIALS

$$ax^2 + bx + c \ , \ a = 1 \quad \text{ie.} \ 1x^2$$

$$x^2 + bx + c = (x + ?)(x + ?)$$ The missing numbers are: **a pair of factors of** c

that **sum to** b

Factorise **fully**:

(1) $x^2 + 5x + 6$ (2) $x^2 - 5x + 6$ (3) $x^2 - 5x - 6$

$1 \times 6 = 2 \times 3 = 6$ $-1,-6 \ or -2,-3$ $-1,6 \ or \ 1,-6 \ or -2,3 \ or \ 2,-3$

$2 + 3 = 5$ $-2 + (-3) = -5$ $1 + (-6) = -5$

use $+2 \ and +3$ *use* $-2 \ and -3$ *use* $+1 \ and -6$

$= (x + 2)(x + 3)$ $= (x - 2)(x - 3)$ $= (x + 1)(x - 6)$

$$ax^2 + bx + c \ , \ a \neq 1 \quad \text{ie. not} \ 1x^2$$

Try out the possible combinations of the factors which could be in the brackets.

Factorise $3t^2 - 10t - 8$

$3 \times (-8) = -24$ *pairs of factors* $\underbrace{1,24 \ or \ 2,12 \ or \ 3,8 \ or \ 4,6}$ *one factor is negative*

$$-12 + 2 = -10$$

$\overbrace{3t^2 - \underline{\mathbf{10t}} - 8}$

factors sum to -10 $-12t + 2t = -10t$

try combinations so that $-12t \ and \ 2t \ are \ obtained$

factor pairs: $3t \times t \ for \ 3t^2$ $1 \times 8 \ or \ 2 \times 4 \ for \ 8 \ , \ one \ factor \ negative$

$3t$ $+1$	$3t$ -8	$3t$ $+2$	$3t$ $+4$
t -8	t $+1$	t -4	t -2
$1t$ $-24t$	$-8t$ $3t$	$2t$ $-12t$ ✓	$4t$ $-6t$

$$\underline{\underline{(3t + 2)(t - 4)}}$$

ALGEBRAIC FRACTIONS

SIMPLIFYING: fully factorise 'top' and 'bottom' and 'cancel' common factors

MULTIPLY/DIVIDE:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \qquad\qquad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

ADD/SUBTRACT: a common denominator is required.

(1) $\dfrac{x^2 - 9}{x^2 + 2x - 3}$

$= \dfrac{(x-3)(x+3)}{(x-1)(x+3)}$ *fully factorise*

$= \dfrac{x-3}{x-1}$ *cancel common factors*

(2) $\dfrac{4}{ab} \div \dfrac{8a}{b^2}$

$= \dfrac{4}{ab} \times \dfrac{b^2}{8a}$

$= \dfrac{4 \times b^2}{ab \times 8a}$

$= \dfrac{4 \times b \times b}{8 \times a \times a \times b}$

$= \dfrac{1 \times b}{2 \times a \times a}$

$= \dfrac{b}{2a^2}$

(3) $\dfrac{x}{2} - \dfrac{x-3}{3}$

$= \dfrac{3x}{6} - \dfrac{2(x-3)}{6}$

$= \dfrac{3x - 2(x-3)}{6}$

$= \dfrac{3x - 2x + 6}{6}$ *notice sign change*

$= \dfrac{x+6}{6}$

(4) $\dfrac{1}{x} + \dfrac{3}{x(x-3)}$

$= \dfrac{1(x-3)}{x(x-3)} + \dfrac{3}{x(x-3)}$

$= \dfrac{x-3+3}{x(x-3)}$

$= \dfrac{x}{x(x-3)}$ *can simplify*

$= \dfrac{1}{x-3}$

CHAPTER 12: EQUATIONS and INEQUALITIES

Simplify by following the rules:

addition and subtraction

$x + a = b$ $x - a = b$

$x \quad = b - a$ $x \quad = b + a$

multiplication and division

$ax = b$ $\dfrac{x}{a} = b$

$x = \dfrac{b}{a}$ $x = ab$

(1) Solve: $5x - 4 = 2x - 19$

$\qquad 3x - 4 = \quad -19$ *subtracted $2x$ from each side*

$\qquad 3x \quad = \quad -15$ *added 4 to each side*

$\qquad \quad x = -5$ *divided each side by 3*

WITH BRACKETS remove first and simplify

(2) Solve: $(4x + 3)(x - 2) = (2x - 3)^2$

$\qquad 4x^2 - 5x - 6 = 4x^2 - 12x + 9$ *removed brackets, fully simplifying*

$\qquad -5x - 6 = \quad -12x + 9$ *subtracted $4x^2$ from each side*

$\qquad 7x - 6 = \quad +9$ *added $12x$ to each side*

$\qquad 7x \quad = \quad 15$ *added 6 to each side*

$\qquad x = \frac{15}{7}$ *divided each side by 7*

WITH FRACTIONS remove first, multiplying by the LCM of the denominators

(3) Solve $\frac{1}{2}(x + 3) + \frac{1}{3}x = 1$

$\qquad \frac{3}{6}(x + 3) + \frac{2}{6}x = \frac{6}{6}$ *write with common denominators*

$\qquad 3(x + 3) + 2x = 6$ *both sides $\times 6$ to remove fractions*

$\qquad 3x + 9 + 2x = 6$

$\qquad 5x = -3$

$\qquad x = -\frac{3}{5}$

INEQUALITIES

Follow the rules for equations, except:

multiply or divide by a **negative** number, reverse the direction of the inequality sign

$$-ax > b \qquad\qquad \frac{x}{-a} > b$$

$$x < \frac{b}{-a} \qquad\qquad x < -ab$$

(1) $8 + 3x > 2$

$$+3x > -6$$

$$x > \frac{-6}{+3} \quad \textit{divided each side by } +3$$
$$\textit{notice sign unchanged}$$

$$x > -2$$

(2) $8 - 3x > 2$

$$-3x > -6 \quad \textit{subtracted 8 from each side}$$

$$x < \frac{-6}{-3} \quad \textit{divided each side by } -3$$
$$\textit{notice sign reversed}$$

$$x < 2 \quad \textit{simplified}$$

RESTRICTIONS ON SOLUTIONS

(1) $x \leq \frac{5}{2}$ *where x is a whole number*

$$x = 0,1,2$$

(2) $-2 \leq x < 2$ *where x is an integer*

$$x = -2,-1,0,1$$

TRANSPOSING FORMULAE (CHANGE OF SUBJECT)

Follow the rules for equations to isolate the **target letter**.

addition and subtraction

$x + a = b$ $\qquad$ $x - a = b$

$x \quad = b - a$ $\qquad$ $x \quad = b + a$

multiplication and division

$ax = b$ $\qquad$ $\dfrac{x}{a} = b$

$x = \dfrac{b}{a}$ $\qquad$ $x = ab$

powers and roots

$x^2 = a$ $\qquad\qquad$ $\sqrt{x} = a$

$x = \sqrt{a}$ $\qquad\qquad$ $x = a^2$

$F = 3r^2 + p$ $\qquad$ Change the subject of the formula to r.

$$r \xrightarrow{\ square\ } r^2 \xrightarrow{\ \times 3\ } 3r^2 \xrightarrow{\ +p\ } F$$

$$\sqrt{\dfrac{F-p}{3}} \xleftarrow{\ \sqrt{\ }\ } \dfrac{F-p}{3} \xleftarrow{\ \div 3\ } F-p \xleftarrow{\ -p\ } F$$

inverse operations in reverse order

$F = 3r^2 + p$

$F - p = 3r^2$ $\qquad$ *subtract p from each side*

$\dfrac{F-p}{3} = r^2$ $\qquad$ *divide each side by 3*

$\sqrt{\dfrac{F-p}{3}} = r$ $\qquad$ *square root both sides*

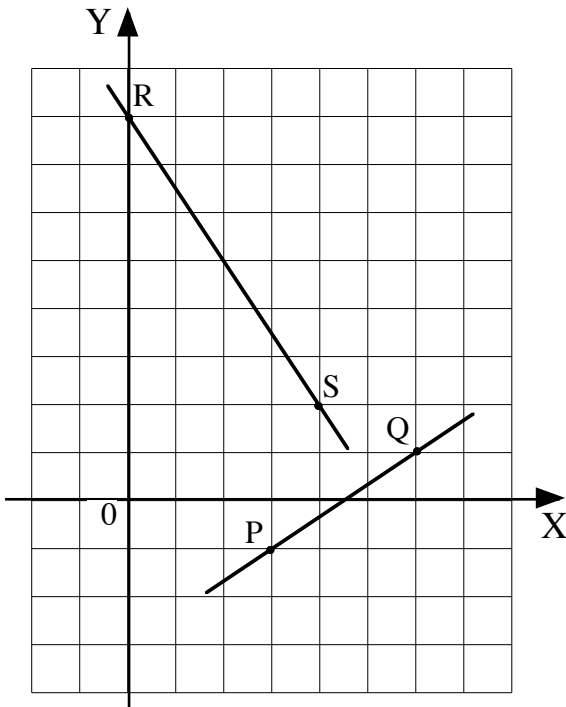$r = \sqrt{\dfrac{F-p}{3}}$ $\qquad$ *subject of formula now r*

CHAPTER 13: STRAIGHT LINE

GRADIENT   The slope of a line is given by the ratio: $m = \dfrac{vertical\ change}{horizontal\ change}$

Using coordinates, the **gradient formula** is $m_{AB} = \dfrac{y_B - y_A}{x_B - x_A}$

$R\,(0,8)$, $S\,(4,2)$

$$m_{RS} = \frac{y_S - y_R}{x_S - x_R} = \frac{2-8}{4-0} = \frac{-6}{4} = -\frac{3}{2}$$

$P\,(3,-1)$, $Q\,(6,1)$

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{1-(-1)}{6-3} = \frac{2}{3}$$

note: same result for $\dfrac{-1-1}{3-6} = \dfrac{-2}{-3} = \dfrac{2}{3}$

EQUATION OF A STRAIGHT LINE

gradient m , y-intercept C units ie. meets the y-axis at (0,*C*) $y = mx + C$

gradient m , through the point (a,*b*) $y - b = m(x - a)$

equation of line RS:

$$m_{RS} = -\tfrac{3}{2} \qquad y = mx + C$$
$$R(0,8)\ \ C = 8 \qquad y = -\tfrac{3}{2}x + 8$$

equation of line PQ:

$$m_{PQ} = \tfrac{2}{3} \qquad\qquad y - b = m(x - a)$$
$$\phantom{m_{PQ}}\ \ \overset{a\ \ \ \ b}{Q(\,6\ ,1\,)} \qquad y - 1 = \tfrac{2}{3}(x - 6)$$
$$\textit{or use point P}$$
$$3y - 3 = 2(x - 6)$$
$$3y - 3 = 2x - 12$$
$$3y = 2x - 9$$

Rearrange the equation to $y = mx + C$ for the gradient and y-intercept.

$$3x + 2y - 16 = 0$$

$$2y = -3x + 16 \qquad \textit{isolate y - term}$$

$$y = -\frac{3}{2}x + 8 \qquad \textit{obtain 1y =}$$

$$y = mx + C \qquad \textit{compare to the general equation}$$

$$m = -\frac{3}{2} , \quad C = 8 \qquad \textit{meets the Y - axis at (0,8)}$$

RATE OF CHANGE The gradient is the rate of change.

distance/time graphs: the gradient is the speed.
speed/time graphs: the gradient is the acceleration.

The steeper the graph, the greater the rate of change.
A positive gradient is an increase, a negative gradient is a decrease.
Gradient zero (horizontal line), no change.

RENTAL TAXIS requires a £4 payment plus a charge of 50p per mile.
CITY TAXIS   charges £1.50 per mile.
Advise on the better buy.
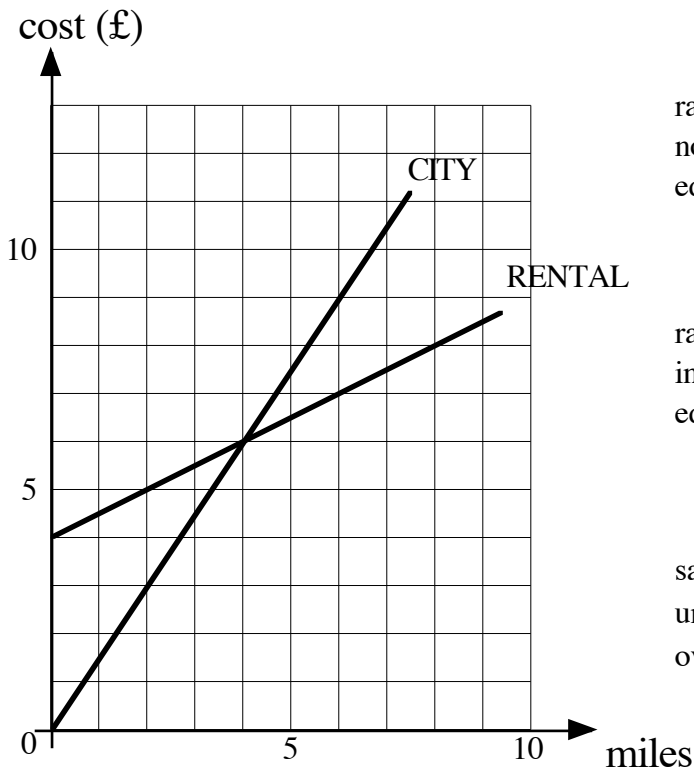
rate of change £1·50 per mile ,   m = 1·5
no initial charge ,   C = 0
equation $y = 1 \cdot 5x$

rate of change £0·50 per mile ,   m = 0·5
initial charge £4 ,   C = 4
equation $y = 0 \cdot 5x + 4$

same cost for 4 miles, but
under 4 miles CITY cheaper
over   4 miles RENTAL cheaper

SCATTER DIAGRAM: LINE OF BEST FIT

W grams

A salt is dissolved in a litre of solvent.

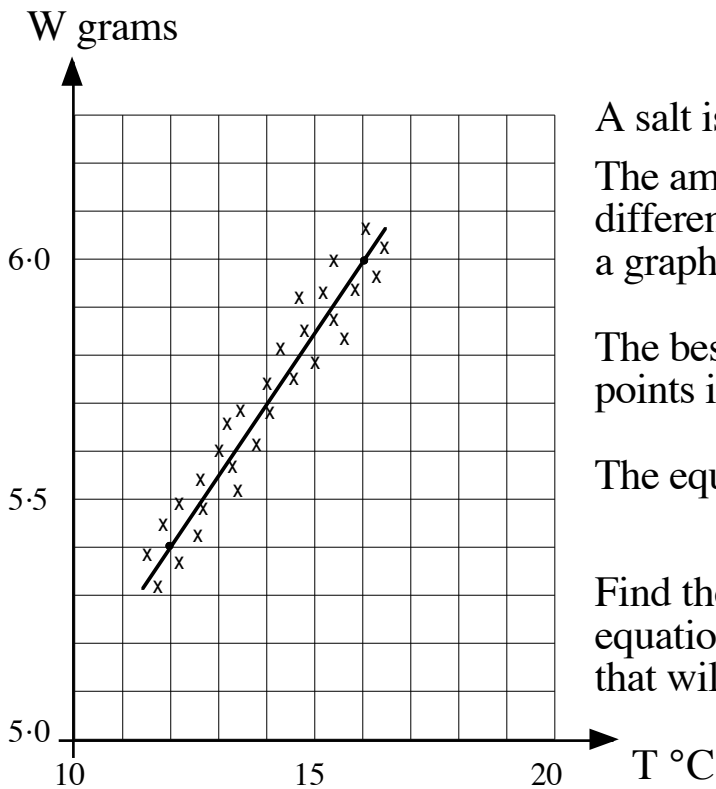The amount of salt that dissolves at different temperatures is recorded and a graph plotted.

The best-fitting straight line through the points is drawn.

The equation of the graph is of the form

$$W = mT + C.$$

Find the equation of the line and use the equation to calculate the mass of salt that will dissolve at 30 °C.

T °C

using two well-separated points on the line $(16, 6 \cdot 0)$ $(12, 5 \cdot 4)$

$$m = \frac{6 \cdot 0 - 5 \cdot 4}{16 - 12} = \frac{0 \cdot 6}{4} = 0 \cdot 15$$

substituting for one point on the line $(16, 6 \cdot 0)$
$$a \quad b$$

$$y - b = m(x - a)$$

$$y - 6 \cdot 0 = 0 \cdot 15(x - 16)$$

$$y - 6 \cdot 0 = 0 \cdot 15 x - 2 \cdot 4$$

$$y = 0 \cdot 15 x + 3 \cdot 6$$

$$\underline{\underline{W = 0 \cdot 15 T + 3 \cdot 6}}$$

$$T = 30 \qquad W = 0 \cdot 15 \times 30 + 3 \cdot 6$$

$$= 4 \cdot 5 + 3 \cdot 6$$

$$= 8 \cdot 1$$

$$\underline{\underline{8 \cdot 1 \ grams}}$$

CHAPTER 14: SIMULTANEOUS EQUATIONS

The equation of a line is a rule connecting the x and y coordinates of any point on the line.

For example,
$$\begin{array}{cc} x & y \\ (-2,12) \end{array} \qquad \begin{array}{c} y + 2x = 8 \\ 12 + 2 \times (-2) = 8 \end{array}$$

To sketch a line, find the points where the line meets the axes.

SOLVE SIMULTANEOUS EQUATIONS: GRAPHICAL METHOD

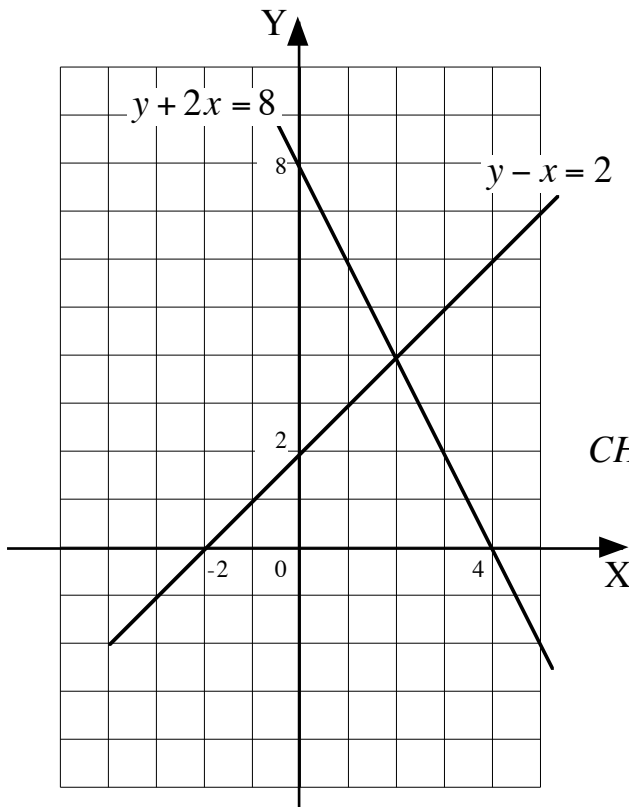Sketch the two lines and the point of intersection is the solution.

Solve **graphically** the system of equations: $y + 2x = 8$
$$y - x = 2$$

$y + 2x = 8$ (1)	$y - x = 2$ (2)
$y + 2 \times 0 = 8$ *substituted for* $x = 0$	$y - 0 = 2$ *substituted for* $x = 0$
$y = 8$ *plot* $(0,8)$	$y = 2$ *plot* $(0,2)$

$y + 2x = 8$ (1)	$y - x = 2$ (2)
$0 + 2x = 8$ *substituted for* $y = 0$	$0 - x = 2$ *substituted for* $y = 0$
$x = 4$ *plot* $(4,0)$	$x = -2$ *plot* $(-2,0)$

point of intersection $(2,4)$

SOLUTION: $x = 2$ *and* $y = 4$

CHECK: $x = 2$ *and* $y = 4$

        substituted in both equations

$y + 2x = 8$	(1)	$y - x = 2$	(2)
$4 + 2 \times 2 = 8$		$4 - 2 = 2$	
$8 = 8$		$2 = 2$	

SOLVE SIMULTANEOUS EQUATIONS: SUBSTITUTION METHOD

Rearrange one equation to $y =$ and substitute for y in the other equations
$$(\text{ or } x =)$$

Solve **algebraically** the system of equations: $y + 2x = 8$
$$3y - x = 10$$

$y + 2x = 8$ (1)   *can choose to rearrange to y = or x =*

$\quad y = 8 - 2x$ *choosing y = avoids fractions as* $x = 4 - \frac{1}{2}y$

$3y - x = 10$ (2)

$3(8 - 2x) - x = 10$ *replace y by* $8 - 2x$

$24 - 6x - x = 10$ *solve*

$\quad -7x = -14$

$\quad\quad x = 2$

$y = 8 - 2x$ (1)   *can choose either equation* (1) *or* (2)

$\quad = 8 - 2 \times 2$ *substituted for* $x = 2$

$\quad y = 4$

SOLUTION: $x = 2$ *and* $y = 4$

CHECK:

$3y - x = 10$ (2)   *using the other equation*

$3 \times 4 - 2 = 10$ *substituted for* $x = 2$ *and* $y = 4$

$\quad 10 = 10$

SOLVE SIMULTANEOUS EQUATIONS: ELIMINATION METHOD

Can add or subtract multiples of the equations to eliminate either the *x* or *y* term.

Solve **algebraically** the system of equations: $4x + 3y = 5$

$$5x - 2y = 12$$

$4x + 3y = 5$	(1) $\times 2$	*can choose to eliminate x or y term*
$5x - 2y = 12$	(2) $\times 3$	*choosing y term, LCM (3y, 2y) = 6y*
$8x + 6y = 10$	(3)	*multiplied each term of (1) by 2 for +6y*
$15x - 6y = 36$	(4)	*multiplied each term of (2) by 3 for −6y*
$23x \qquad = 46$	(3) + (4)	*added equations, adding "like" terms*
$x = 2$		*+6y and −6y added to 0 (ie eliminate)*
$4x + 3y = 5$	(1)	*can choose either equation (1) or (2)*
$4 \times 2 + 3y = 5$		*substituted for x = 2*
$8 + 3y = 5$		
$3y = -3$		
$y = -1$		

SOLUTION: $\quad x = 2 \ and \ y = -1$

CHECK:

$5x - 2y = 12$ (2) $\qquad$ *using the other equation*

$5 \times 2 - 2 \times (-1) = 12$ $\qquad$ *substituted for x = 2 and y = −1*

$10 - (-2) = 12$

$12 = 12$

CHAPTER 15: QUADRATIC EQUATIONS

A function pairs one number with another, its IMAGE. It can be defined by a formula.
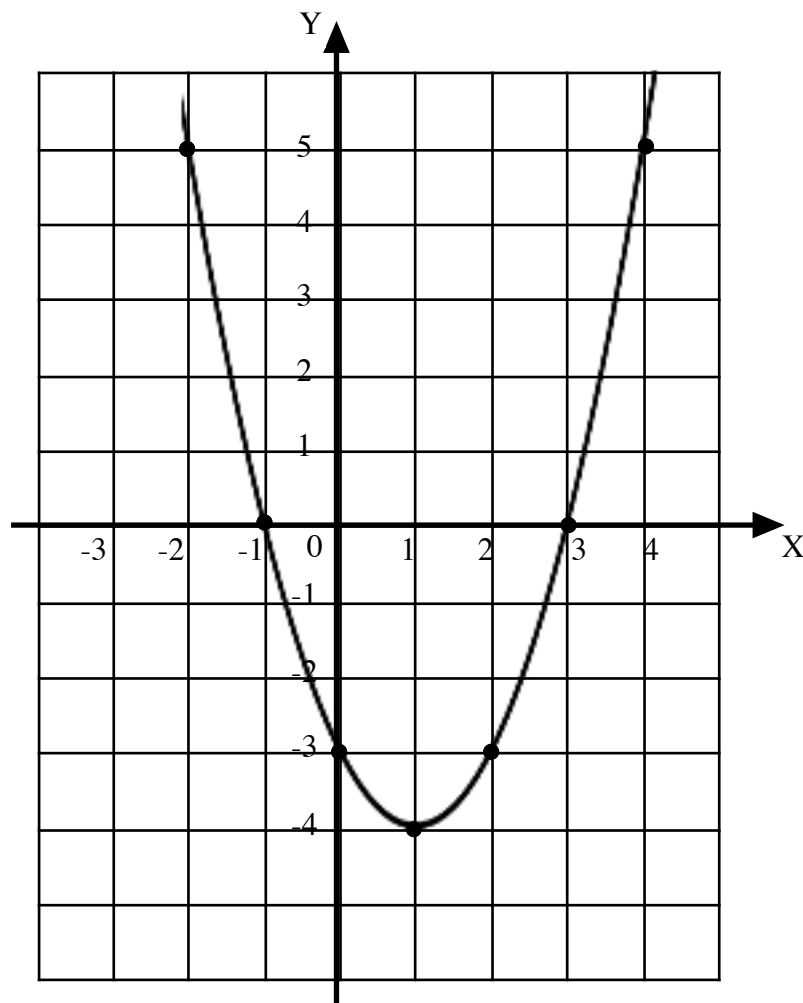
QUADRATIC FUNCTIONS

$f(x) = ax^2 + bx + c$, $a \neq 0$, where a, b and c are constants.

$f(x) = x^2 - 2x - 3$

$f(-2) = (-2)^2 - 2 \times (-2) - 3 = 4 + 4 - 3 = 5$, the image of -2 is 5.

x	-2	-1	0	1	2	3	4
$f(x)$	5	0	-3	-4	-3	0	5
points	(-2,5)	(-1,0)	(0,-3)	(1,-4)	(2,-3)	(3,0)	(4,5)

If all possible values of x are considered, a graph will show the images (the RANGE).
The graph is a symmetrical curve called a PARABOLA.

The graph meets the x-axis where $x^2 - 2x - 3 = 0$.
The **zeros** of the graph are -1 and 3 which are the **roots** of the equation.

QUADRATIC EQUATIONS

An equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, where a, b and c are constants.

The values of x that satisfy the equation are the **roots** of the equation.
The quadratic formula can be used to solve the equation.

QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad , a \neq 0$$

Find the **roots** of the equation $3x^2 - 5x - 1 = 0$, correct to two decimal places.

$3x^2 - 5x - 1 = 0$
$ax^2 + bx + c = 0$

$a = 3$, $b = -5$, $c = -1$

$b^2 - 4ac = (-5)^2 - 4 \times 3 \times (-1) = 37$

$-b = -(-5) = +5$
$2a = 2 \times 3 = 6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{5 \pm \sqrt{37}}{6}$$

$$= \frac{5 - \sqrt{37}}{6} \quad or \quad \frac{5 + \sqrt{37}}{6}$$

$$= \frac{-1 \cdot 0827....}{6} \quad or \quad \frac{11 \cdot 0827....}{6}$$

$$x = -0 \cdot 1804.... \quad or \quad 1 \cdot 8471....$$

$$\underline{\text{roots are } -0 \cdot 18 \text{ and } 1 \cdot 85}$$
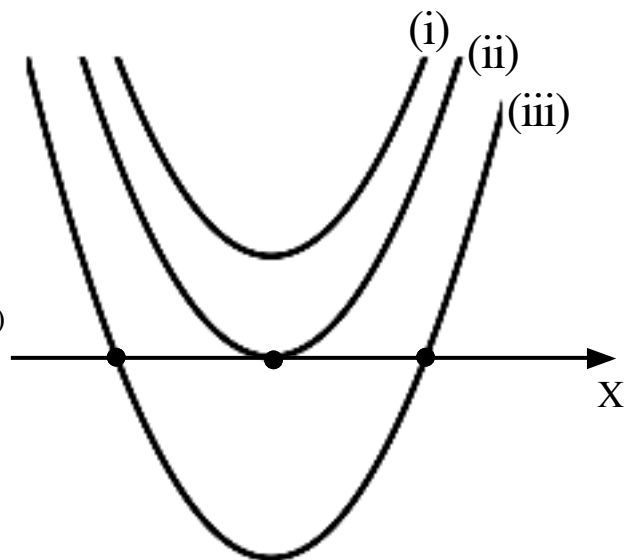
DISCRIMINANT
$b^2 - 4ac$ gives the nature of the roots

(i) $b^2 - 4ac < 0$ NO REAL ROOTS
   ie. negative

(ii) $b^2 - 4ac = 0$ ONE REAL ROOT (repeated)
   (two EQUAL roots)

(iii) $b^2 - 4ac > 0$ TWO REAL ROOTS
   ie. positive   (and DISTINCT)

page 34

Find k so that $2x^2 - 4x - k = 0$ has **real** roots.

$2x^2 - 4x - k = 0$ *for real roots ,* $b^2 - 4ac \geq 0$

$ax^2 + bx + c = 0$ $16 + 8k \geq 0$

$a = 2$, $b = -4$, $c = -k$ $8k \geq -16$

$b^2 - 4ac = (-4)^2 - 4 \times 2 \times (-k)$ $k \geq -2$

$\qquad\qquad = 16 + 8k$

FACTORISATION

RATIONAL ROOTS $b^2 - 4ac =$ a square number   ie. $0, 1, 4, 9, 16$.....
(non-surds)   the quadratic can be factorised to solve the equation.

IRRATIONAL ROOTS $b^2 - 4ac \neq$ a square number
(surds)   solve the equation by formula.

Solve:

(1) $4n - 2n^2 = 0$

$\quad$ $2n(2 - n) = 0$

$\quad$ $2n = 0$ or $2 - n = 0$

$\quad$ $n = 0$ or $n = 2$

(2) $2t^2 + t - 6 = 0$

$\quad$ $(2t - 3)(t + 2) = 0$

$\quad$ $2t - 3 = 0$ or $t + 2 = 0$

$\quad$ $2t = 3$

$\quad$ $t = \dfrac{3}{2}$ or $t = -2$

The equation may need to be rearranged:

(3) $(w + 1)^2 = 2(w + 5)$

$\quad$ $w^2 + 2w + 1 = 2w + 10$

$\quad$ $w^2 - 9 = 0$

$\quad$ $(w + 3)(w - 3) = 0$

$\quad$ $w + 3 = 0$ or $w - 3 = 0$

$\quad$ $w = -3$ or $w = 3$

(4) $x + 2 = \dfrac{15}{x}$, $x \neq 0$

$\quad$ $x(x + 2) = 15$

$\quad$ $x^2 + 2x = 15$

$\quad$ $x^2 + 2x - 15 = 0$

$\quad$ $(x + 5)(x - 3) = 0$

$\quad$ $x + 5 = 0$ or $x - 3 = 0$

$\quad$ $x = -5$ or $x = 3$

GRAPHS

A sketch of the graph of a quadratic function should show where the parabola meets the axes and the maximum or minimum turning point.

Sketch the graph $y = x^2 - 2x - 3$.

(i) *meets the Y-axis where x = 0*

$y = 0^2 - 2 \times 0 - 3 = -3$ *point (0,-3)*

(ii) *meets the X-axis where y = 0*
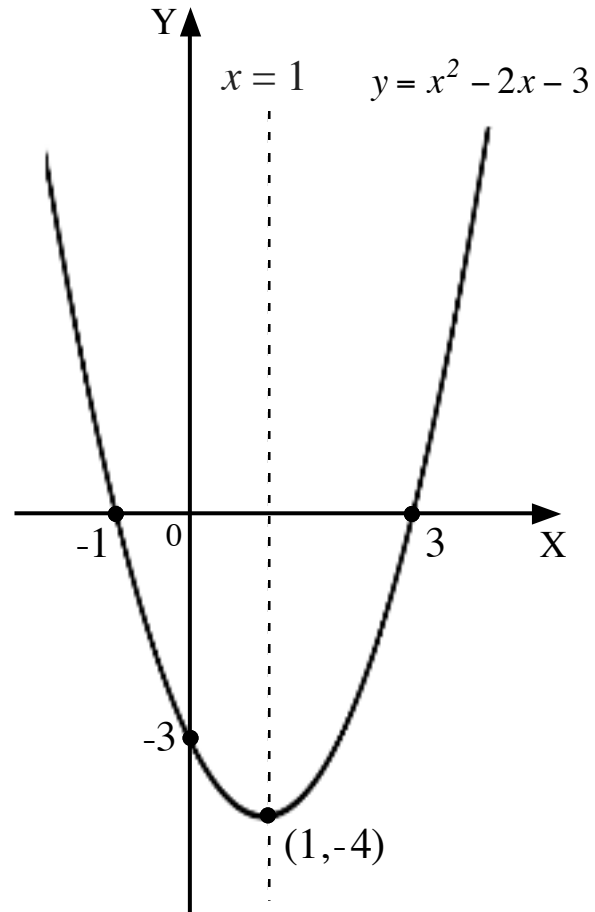
$$x^2 - 2x - 3 = 0$$
$$(x + 1)(x - 3) = 0$$
$$x = -1 \quad \text{or} \quad x = 3$$
points (-1,0) and (3,0)

(iii) axis of symmetry

vertical line half-way between the zeros

$x = \dfrac{-1 + 3}{2} = \dfrac{2}{2} = 1$, *equation* $x = 1$.

(iv) turning point

lies on the axis of symmetry $x = 1$

$y = 1^2 - 2 \times 1 - 3 = -4$ *point (1,-4)*

Note: (1) from the graph, (1,-4) is a **minimum turning point**.

      (2) the **minimum value** of the function is -4.

APPLICATIONS

Problems involving maxima or minima which can be modelled by a quadratic equation.

A sheet of metal 40 cm. wide is folded x cm from each end to form a gutter.
To maximise water flow the rectangular cross-section should be as large as possible.
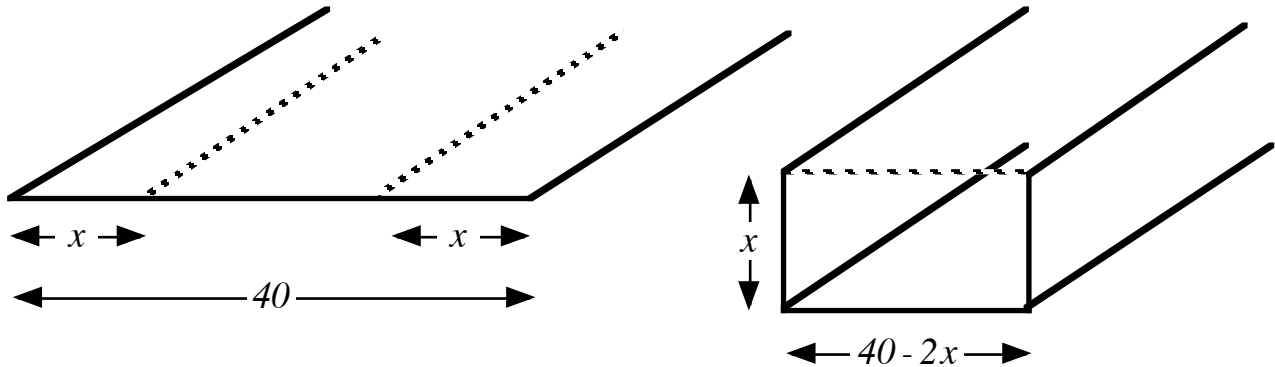
Find the maximum cross-sectional area possible.

$A = lb$
$\quad = x(40 - 2x) \qquad$ *sketch the graph* $A = 40x - 2x^2$
$\quad = 40x - 2x^2$

Zeros:

$40x - 2x^2 = 0$
$2x(20 - x) = 0$
$2x = 0 \quad$ or $\quad 20 - x = 0$
$\quad x = 0 \quad$ or $\qquad x = 20$

Turning Point:

$(0 + 20) \div 2 = 10$
axis of symmetry $x = 10$

$y = 40x - 2x^2$
$y = 40 \times 10 - 2 \times 10^2 = 200$

maximum turning point(10,200)
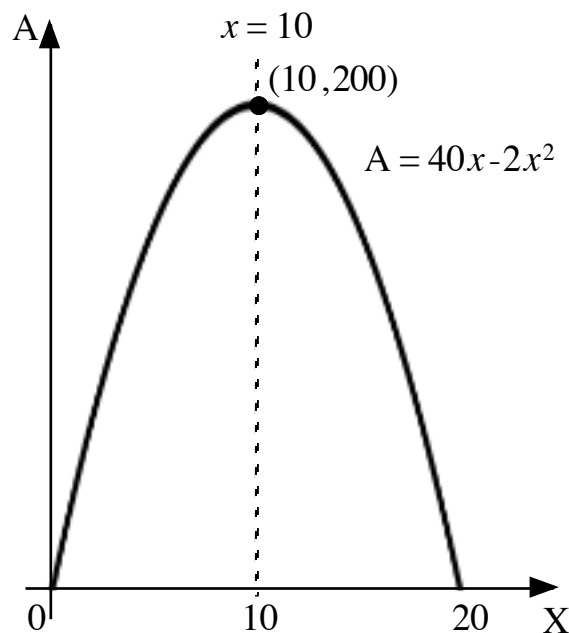
Maximum area 200 square centimetres.

EQUATION OF A GRAPH

$$y = k(x - a)(x - b)$$

a and b are the zeros of the graph

k is a constant.
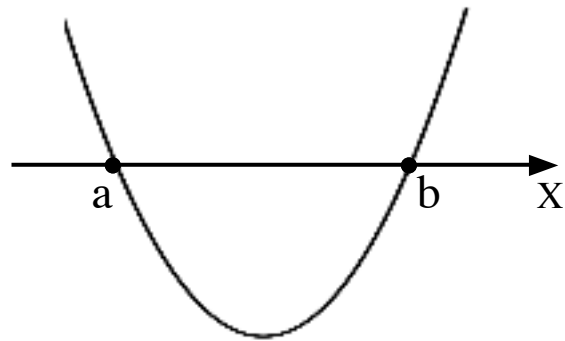
Write the equation of the graph.

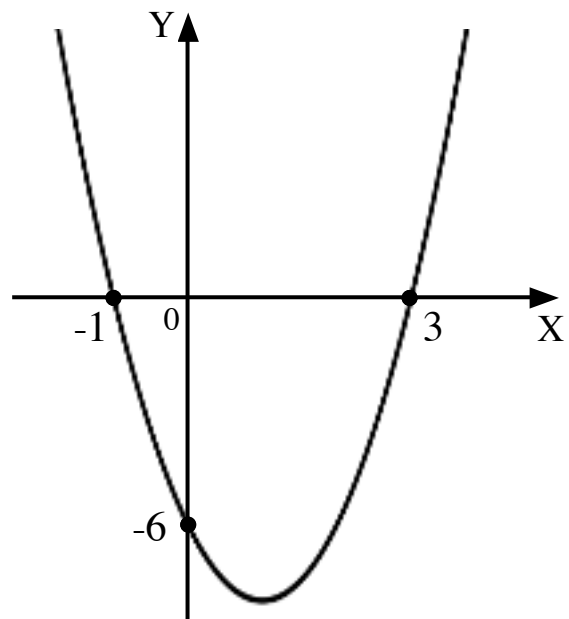$$y = k(x - a)(x - b)$$

zeros -1 and 3

$$y = k(x + 1)(x - 3)$$

$$\begin{array}{cc} x & y \end{array}$$
for $(0,-6)$, $-6 = k(0 + 1)(0 - 3)$

$$-6 = k \times 1 \times (-3)$$

$$-6 = -3k$$

$$k = 2$$

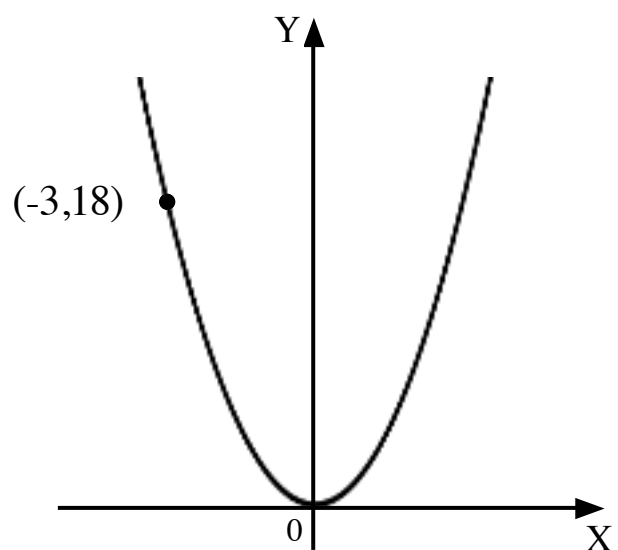$$y = 2(x + 1)(x - 3)$$

Write the equation of the graph.

$$y = kx^2$$

$$\begin{array}{cc} x & y \end{array}$$
for $(-3,18)$, $18 = k \times (-3)^2$

$$18 = 9k$$

$$k = 2$$

$$y = 2x^2$$

(-3,18)

COMPLETED SQUARE:

Quadratic functions written in the form $y = \pm 1(x-a)^2 + b$, a and b are constants.

> axis of symmetry $\quad x = a$
> turning point $\qquad (a,b)$, minimum for $+1$, maximum for -1

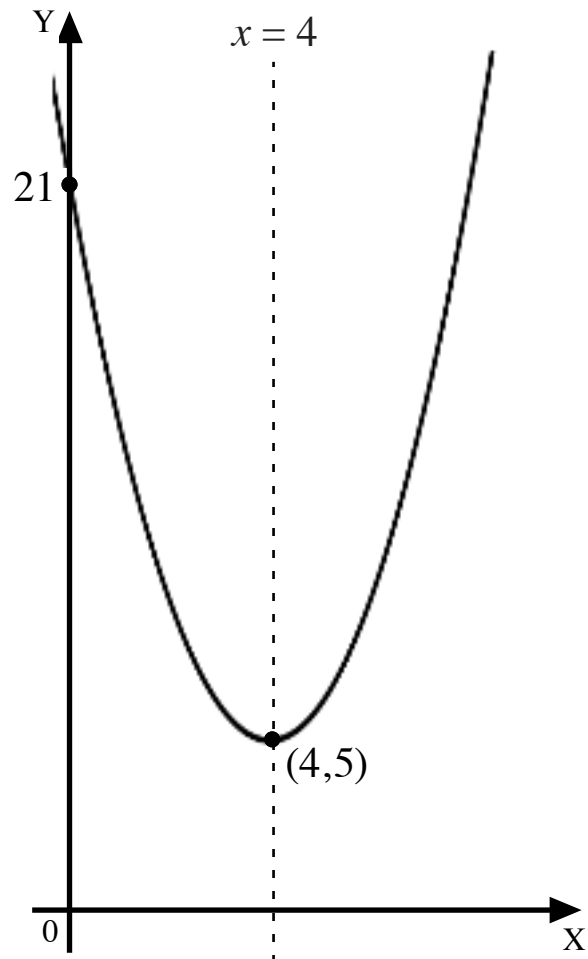$y = x^2 - 8x + 21$ can be written as $y = (x-4)^2 + 5$

meets the y-axis where $x = 0$

$y = (0-4)^2 + 5 = 16 + 5 = 21$

point $(0,21)$

turning point: $y = +1(x-4)^2 + 5$

minimum turning point $(4,5)$

axis of symmetry $\quad x = 4$

COMPLETING THE SQUARE:

$x^2 - 8x + 21$

$x^2 - 8x \qquad\qquad + 21$ *coefficient of x, -8, is halved and squared, $(-4)^2$ to 16*

$= x^2 - 8x + 16 \quad -16 + 21$ *add and subtract 16*

$= \quad (x-4)^2 \quad\quad -16 + 21$ *complete square around trinomial*

$= \quad (x-4)^2 + 5$

page 39

CHAPTER 16: TRIGONOMETRY: RIGHT-ANGLED TRIANGLES

SOH-CAH-TOA   The sides of a right-angled triangle are labelled:
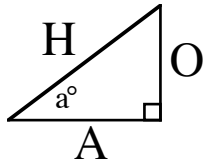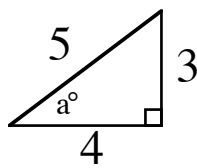

Opposite: opposite the angle a°.
Adjacent: next to the angle a°.
Hypotenuse: opposite the right angle.

The ratios of sides $\dfrac{O}{H}, \dfrac{A}{H}$ and $\dfrac{O}{A}$ have values which depend on the size of angle a°.

These are called the sine, cosine and tangents of a°, written sin a° , cos a° and tan a°.

For example,

$$S = \frac{O}{H} \qquad C = \frac{A}{H} \qquad T = \frac{O}{A}$$

$$\sin a° = \frac{3}{5} \qquad \cos a° = \frac{4}{5} \qquad \tan a° = \frac{3}{4}$$

The trig. function acts on an angle to produce the value of the ratio.
The inverse trig. function acts on the value of a ratio to produce the angle.

For example,

$$\sin 30° = 0.5$$
$$\sin^{-1} 0.5 = 30°$$

ACCURACY
Rounding the angle or the value in a calculation can result in significant errors.

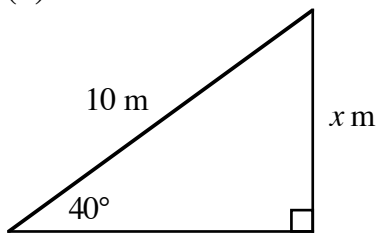For example,

$100 \times \tan 69.5° = 267.462... \approx 267$ $\qquad$ $\tan^{-1} 2.747 = 69.996... \approx 70.0$
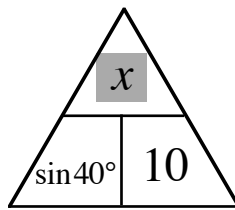
$100 \times \tan 70° = 274.747... \approx 275$ $\qquad$ $\tan^{-1} 2.7 = 69.676... \approx 69.7$

FINDING AN UNKNOWN SIDE

(1) Find x.

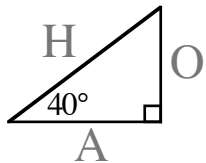$$\sin 40° = \frac{x}{10}$$

rearrange for x

$$x = 10 \times \sin 40°$$ *ensure calculator set to DEGREES*

$$= 6 \cdot 427....$$

$$x = 6 \cdot 4$$

know H , find O

SOH-CAH-TOA

---

(2) Find y.
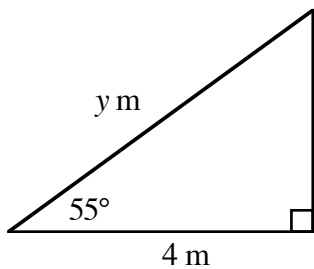
$$\cos 55° = \frac{4}{y}$$

rearrange for y
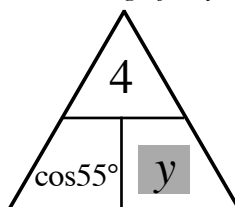
$$y = \frac{4}{\cos 55°}$$ *4 ÷ cos 55°, calculator set to DEGREES*

$$= 6 \cdot 973....$$

$$y = 7 \cdot 0$$
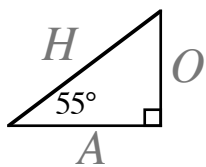
know A , find H

SOH-**CAH**-**TOA**

FINDING AN UNKNOWN ANGLE

(3) Find x.
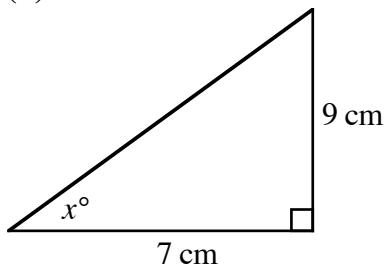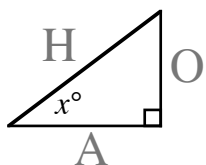
$$\tan x° = \frac{9}{7}$$

$$x = \tan^{-1}\left(\frac{9}{7}\right)$$ *use brackets for (9 ÷ 7), calculator set to DEGREES*

$$= 52 \cdot 125....$$

$$x = 52 \cdot 1$$

know O , know A

SOH-**CAH**-**TOA**

CHAPTER 17: TRIGONOMETRY: GRAPHS & EQUATIONS

The cosine graph is the sine graph shifted 90° to the left.
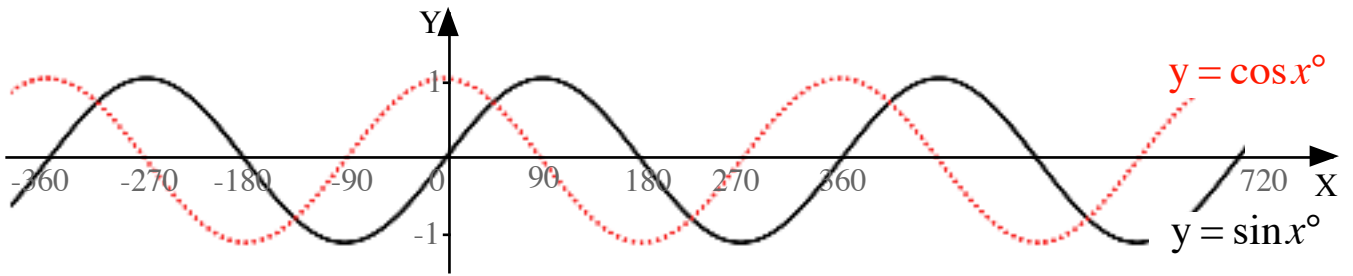
The graphs have a PERIOD of 360° (repeat every 360°).
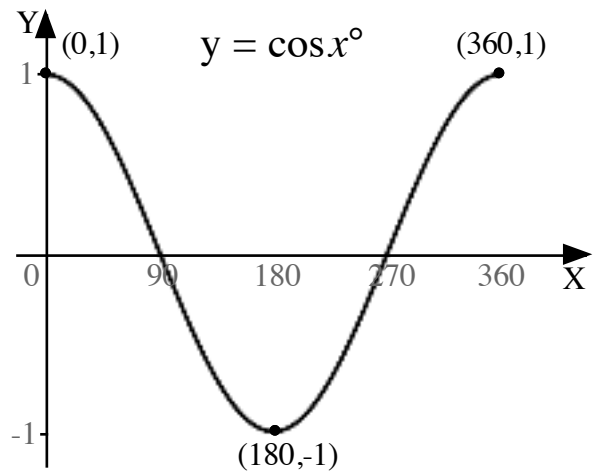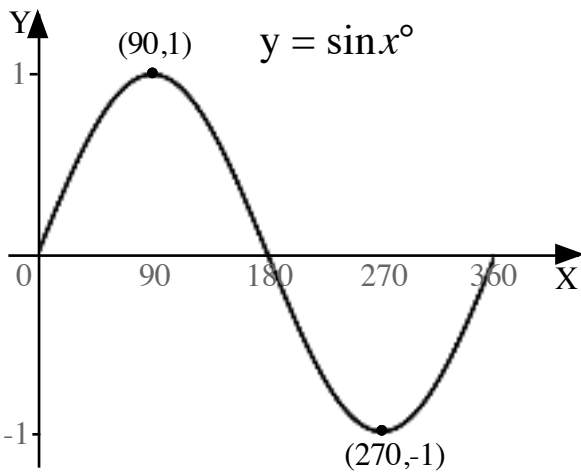
Turning points:
maximum (90,1) , minimum (270,-1)   maximum (0,1) , minimum (180,-1)

$$y = \tan x°$$

The tangent graph has a PERIOD of 180°.

TRANSFORMATIONS   Same rules for $y = \sin x°$ and $y = \cos x°$.

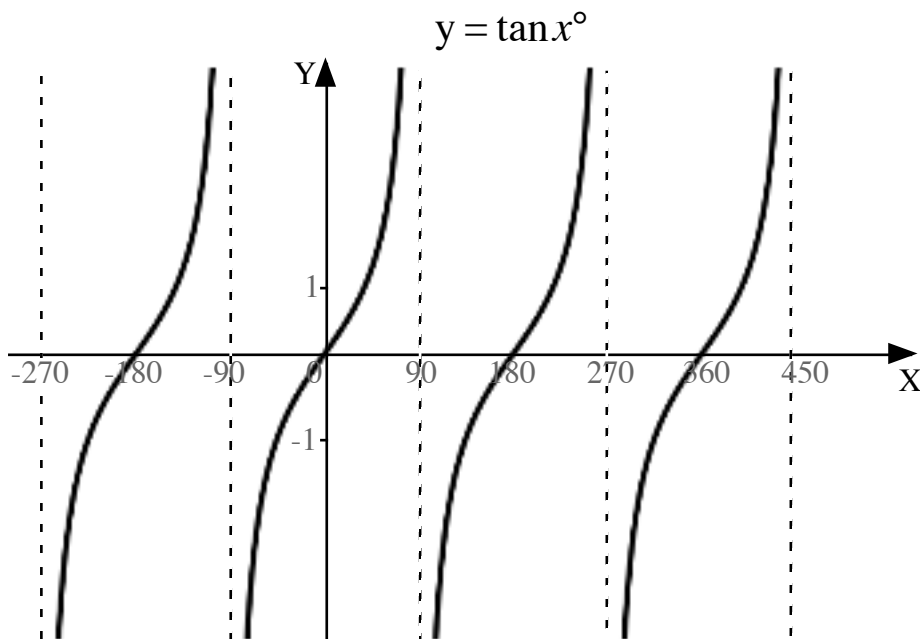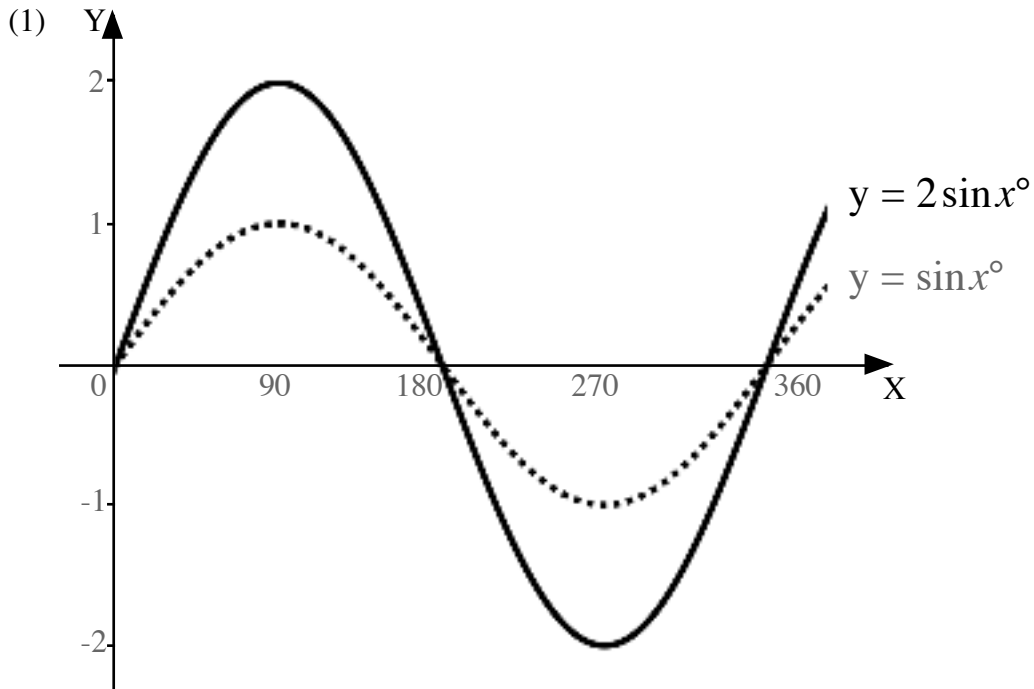Y-STRETCH $y = \mathbf{n}\sin x°$ y-coordinates multiplied by **n**.
                                          amplitude **n** units
                                          maximum value $+\mathbf{n}$, minimum value $-\mathbf{n}$

(1)

$y = 2\sin x°$

$y = \sin x°$

X-STRETCH $y = \sin \mathbf{n}x°$ x-coordinates divided by **n**.
                                          period $360° \div \mathbf{n}$. There are **n** cycles in $360°$.

(2)

$y = \sin 2x°$

$y = \sin x°$

period 180° , 2 cycles in 360°

page 43
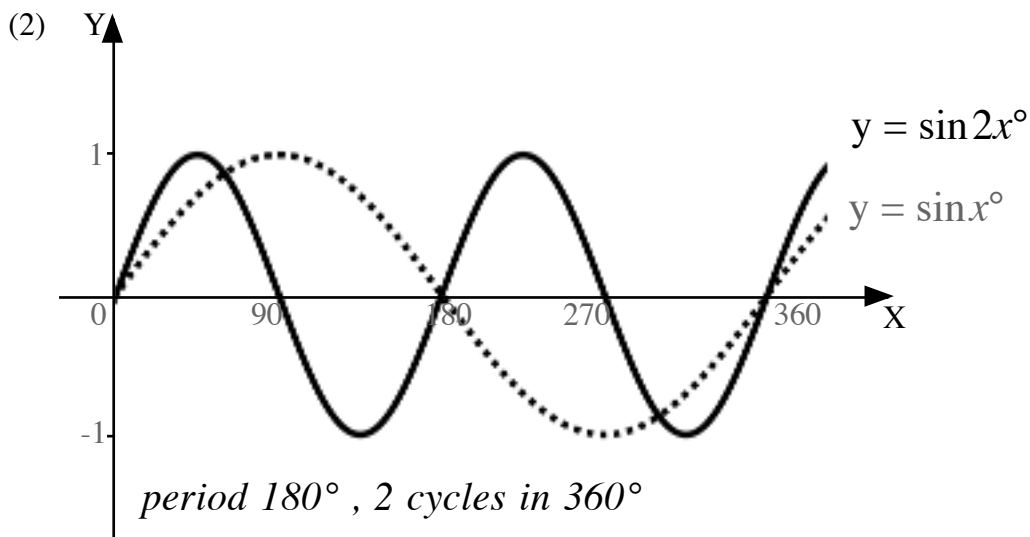
Y-SHIFT $\qquad$ $y = \sin x° + n$ $\qquad$ add **n** units to y-coordinates

graph shifted **n** units vertically.

(3)

$$y = \sin x° + 2$$

$$y = \sin x°$$
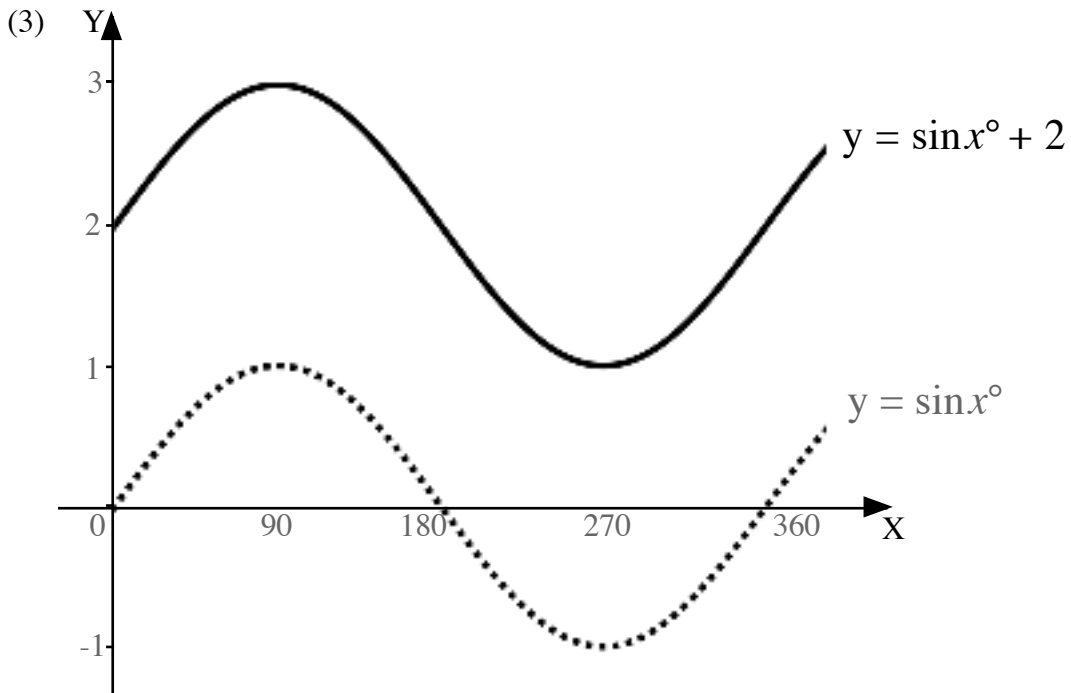
X-SHIFT $\qquad$ $y = \sin(x + n)°$ $\qquad$ subtract **n** units from the x-coordinates

graph shifted **-n°** horizontally.

(4)

$$y = \sin x°$$
$$y = \sin(x - 20)°$$

graph shifted 20° to the right
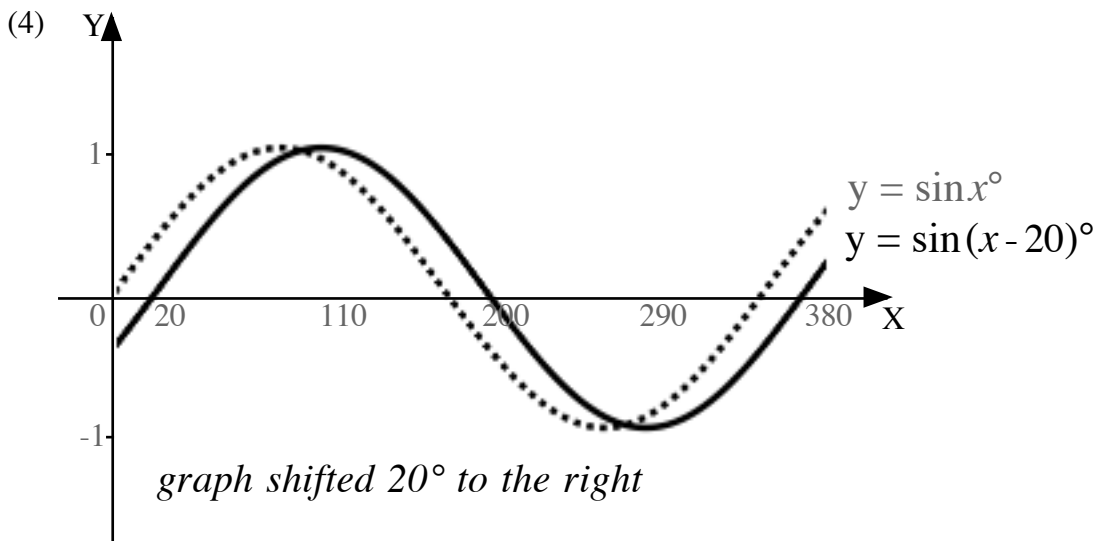
NOTE: for $y = \sin(x + 20)°$ the graph $y = \sin x°$ would be shifted 20° to the left.

COMBINING TRANSFORMATIONS

(1)

$$y = 7\sin 3x° + 8$$

amplitude 7

oscillates around the line y = 8

period 120°

by   90÷3   180÷3   270÷3   360÷3

Turning points: maximum (30,15) , minimum (90,1)

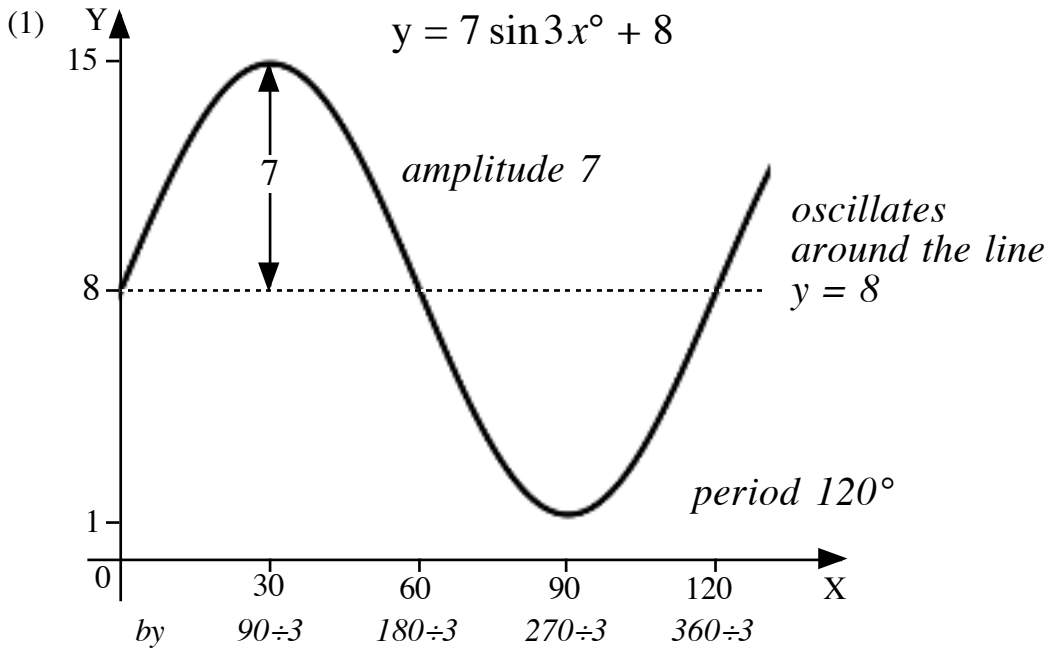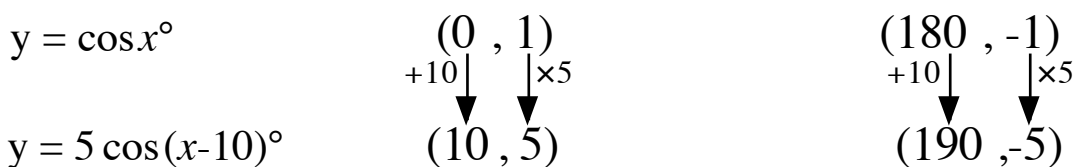$y = \sin x°$ (90 , 1)   (270 , -1)

                    ÷3 | | ×7 + 8   ÷3 | | ×7 + 8

$y = 7\sin 3x° + 8$ (30 ,15)   (90 , 1)

(2)

$$y = 5\cos(x - 10)°$$

period 360°

Turning points: maximum (10,5) , minimum (190,-5)

$y = \cos x°$ (0 , 1)   (180 , -1)

                    +10 | | ×5   +10 | | ×5

$y = 5\cos(x\text{-}10)°$ (10 , 5)   (190 ,-5)
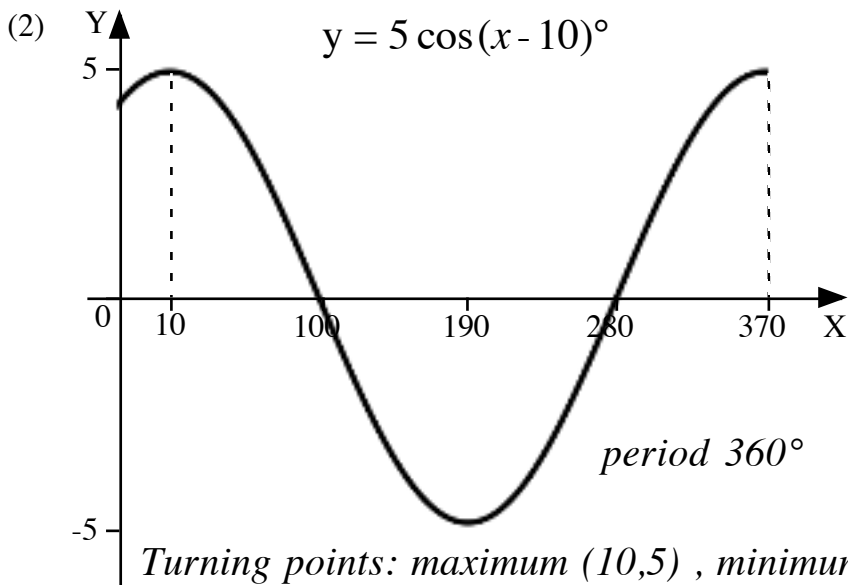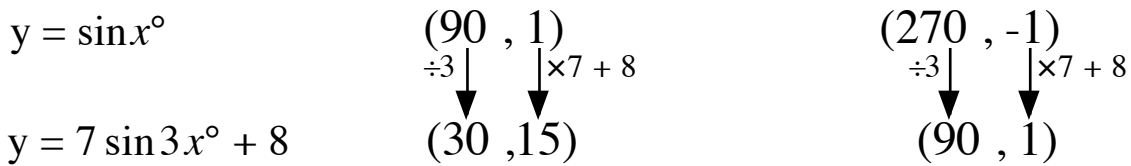
EQUATIONS

The graphs with equations $y = 5 + 3\cos x°$ and $y = 4$ are shown.

Find the x coordinates of the points of intersection A and B.

$y = 5 + 3\cos x°$

$y = 4$
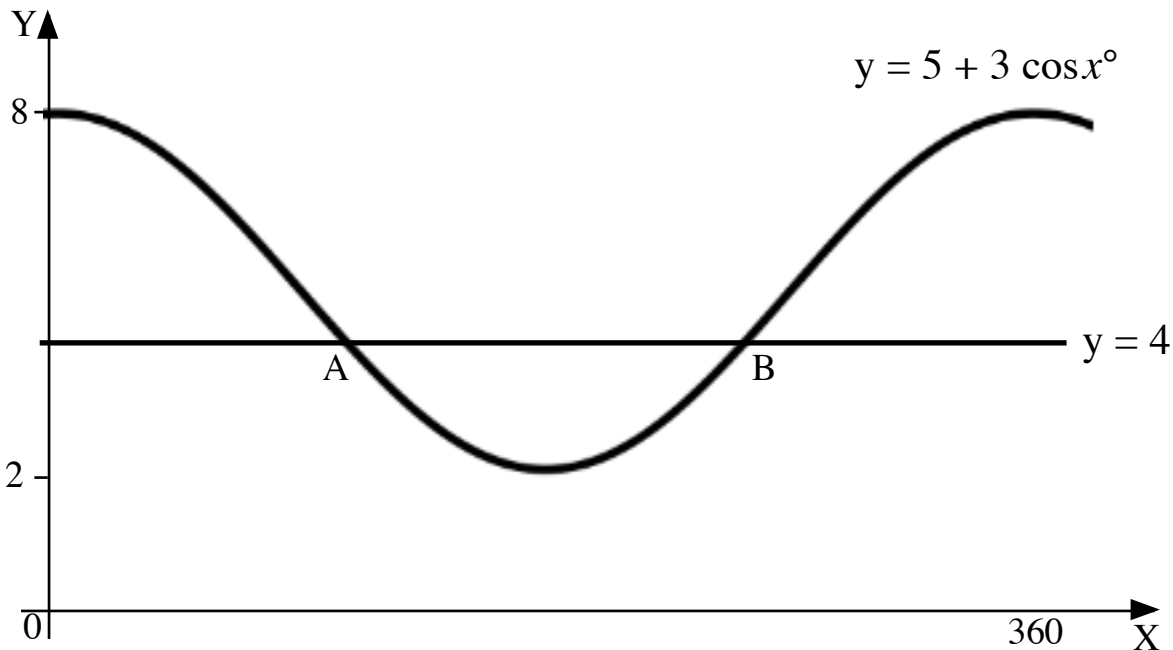
$5 + 3\cos x° = 4$

$3\cos x° = -1$

$\cos x° = -\dfrac{1}{3}$

$x = 109{\cdot}5 \ \ or \ \ 250{\cdot}5$

* A, S, T, C is where functions are **positive**:

	S	A	
	cosine negative ✓	*cosine positive* ✗	
	180 - a = 109·5	a = cos $^{-1}$ $^1/_3$ = 70·528...	
	180 + a = 250·5	360 - a = 289·5	
	cosine negative ✓	*cosine positive* ✗	
	T	C	

* A   all functions are positive
  S   sine function only is positive
  T   tangent function only is positive
  C   cosine function only is positive

IDENTITIES

$$\sin^2 x° + \cos^2 x° = 1$$

$$\tan x° = \frac{\sin x°}{\cos x°}$$

Simplify $\dfrac{1 - \cos^2 x°}{\sin x° \cos x°}$.

$$= \frac{\sin^2 x°}{\sin x° \cos x°}$$ *since* $\sin^2 x° + \cos^2 x° = 1$

$$\sin^2 x° = 1 - \cos^2 x°$$

$$= \frac{\sin x° \sin x°}{\sin x° \cos x°}$$ *"cancel"* $\sin x°$

$$= \frac{\sin x°}{\cos x°}$$

$$= \tan x°$$

EXACT VALUES
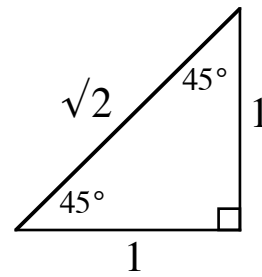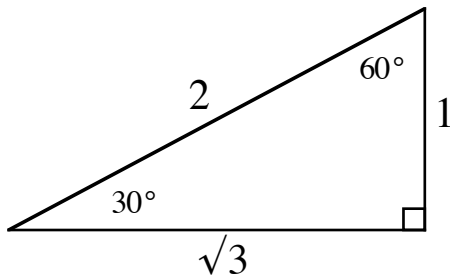
For example,

$$\sin 60° = \frac{\sqrt{3}}{2} \quad , \quad \tan 30° = \frac{1}{\sqrt{3}}$$

$$\cos 45° = \frac{1}{\sqrt{2}} \quad , \quad \tan 45° = \frac{1}{1} = 1$$

ANGLES > 90°

$$\sin 240°$$

$$= \sin(180° + 60°)$$

$$= -\sin 60°$$

$$= -\frac{\sqrt{3}}{2}$$

S | A

180 - a   a = 60

240°

60°

180 + a = 240   360 - a

T | C

sine negative

CHAPTER 18: TRIGONOMETRY: TRIANGLE FORMULAE

SINE RULE

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
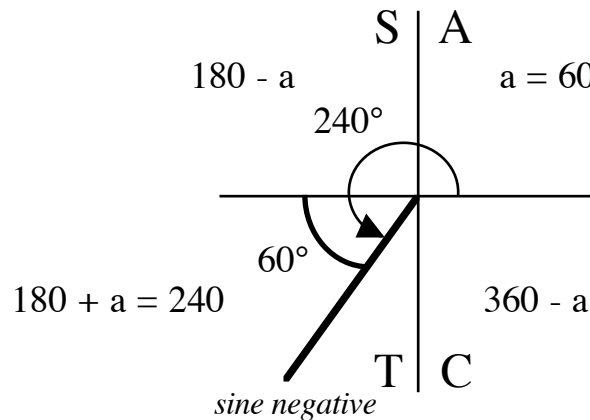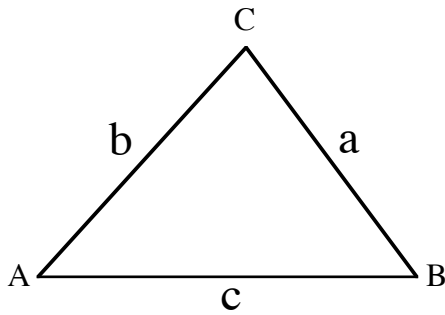
NOTE: requires at least one side and its opposite angle to be known.

FINDING AN UNKNOWN SIDE
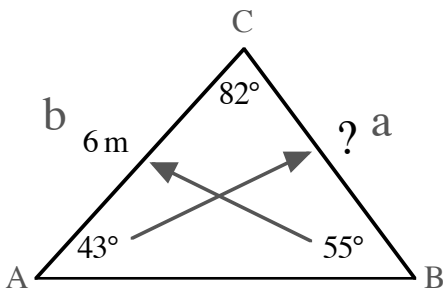
Find the length of side BC

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 43°} = \frac{6}{\sin 55°}$$

$$a = \frac{6}{\sin 55°} \times \sin 43°$$

$$= 4 \cdot 995.....$$

$$BC \approx 5 \cdot 0 \; m$$
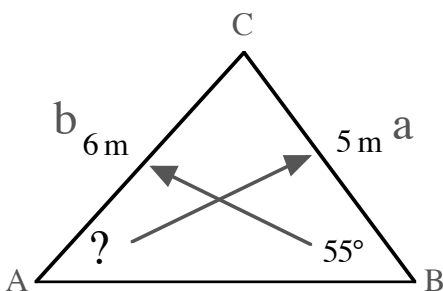
FINDING AN UNKNOWN ANGLE

Find the size of angle BAC.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{5} = \frac{\sin 55°}{6}$$

$$\sin A = \frac{\sin 55°}{6} \times 5$$

$$= 0 \cdot 682.....$$

$$A = \sin^{-1} 0 \cdot 682.....$$

$$= 43 \cdot 049.....$$

$$\angle BAC \approx 43.0°$$

COSINE RULE

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

FINDING AN UNKNOWN SIDE

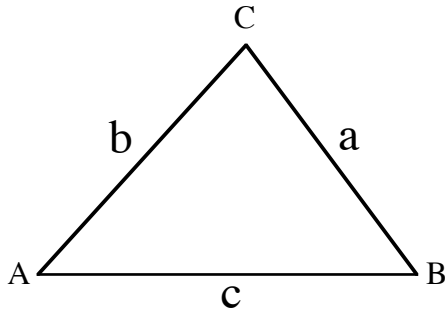NOTE: requires knowing 2 sides and the angle between them.

Find the length of side BC.
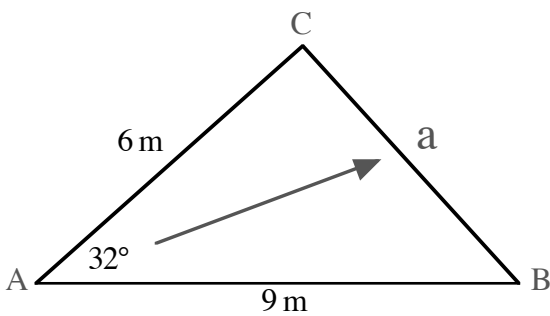
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$= 6^2 + 9^2 - 2 \times 6 \times 9 \times \cos 32°$$
$$a^2 = 25 \cdot 410.....$$
$$a = \sqrt{25 \cdot 410.....}$$
$$= 5 \cdot 040.....$$
$$BC \approx 5 \cdot 0 \ m$$

FINDING AN UNKNOWN ANGLE

NOTE: requires knowing all 3 sides.

Find the size of angle BAC.
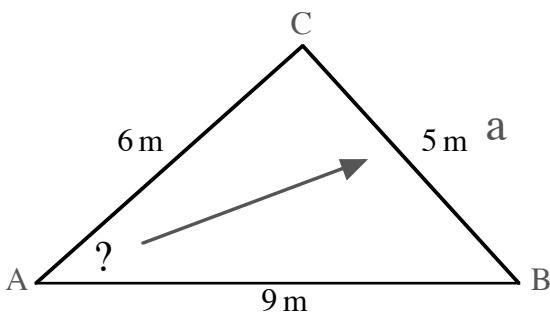
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{6^2 + 9^2 - 5^2}{2 \times 6 \times 9}$$
$$\cos A = 0 \cdot 85185.....$$
$$A = \cos^{-1}(0 \cdot 85185.....)$$
$$= 31.586.....$$
$$\angle BAC \approx 31 \cdot 6°$$

AREA FORMULA
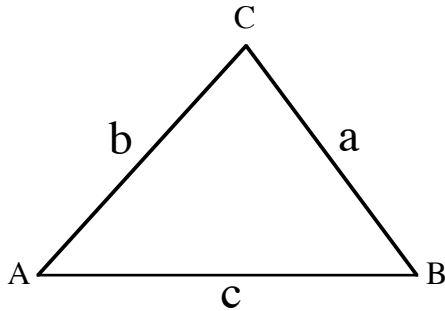
$$Area \, \triangle ABC = \frac{1}{2}ab \sin C$$

NOTE: requires knowing 2 sides and the angle between them.

(1)

Find the area of the triangle.
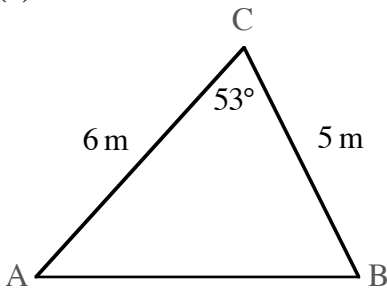
$$Area \, \triangle ABC = \frac{1}{2}ab \sin C$$
$$= \frac{1}{2} \times 5 \times 6 \times \sin 53°$$
$$= 11·979.....$$
$$Area \approx 12·0 \, m^2$$

(2)
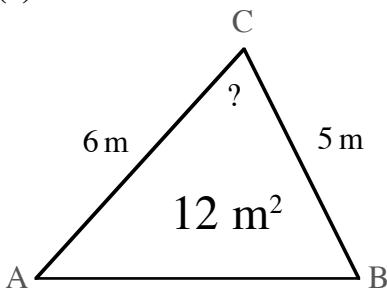
Find angle ACB.

$$Area \, \triangle ABC = \frac{1}{2}ab \sin C$$
$$12 = \frac{1}{2} \times 5 \times 6 \times \sin C \qquad double \, both \, sides$$
$$24 = 30 \times \sin C$$
$$\sin C = 24 \div 30 = 0·8$$
$$C = \sin^{-1}(0.8)$$
$$= 53.130....° \quad or \quad 126.869....°$$

(*from* $180° - 53.130...°$
as angle could be obtuse)

$$\angle ACB \approx 53·1° \qquad from \, diagram, \, angle \, acute$$

CHAPTER 19: VECTORS

SCALAR quantities have size(magnitude). eg. time, speed, volume

VECTOR quantities have **size** and **direction**. eg. force, velocity

A **directed line segment** represents a vector.

Vectors can be written in component form as **column vectors**
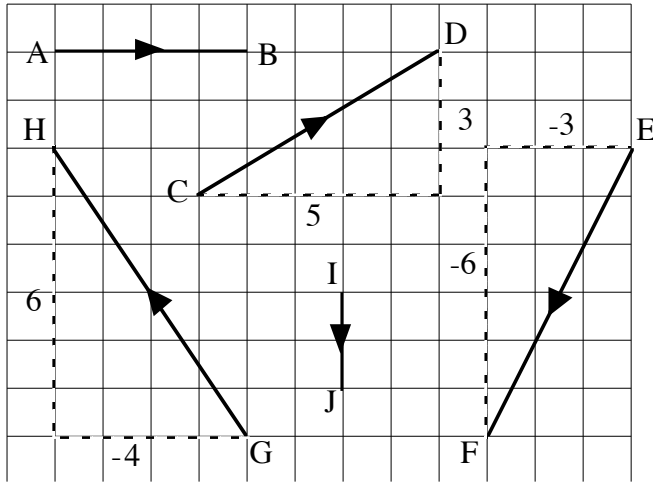
$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \qquad \overrightarrow{CD} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \qquad \overrightarrow{EF} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$$

$$\overrightarrow{GH} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} \qquad \overrightarrow{IJ} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

negative vector, opposite direction

$$\overrightarrow{HG} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

SIZE follows from Pyth. Thm

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |\mathbf{u}| = \sqrt{a^2 + b^2} \qquad |\overrightarrow{GH}| = \sqrt{(-4)^2 + 6^2} = \sqrt{52} = 2\sqrt{13} \ units$$

parallelogram
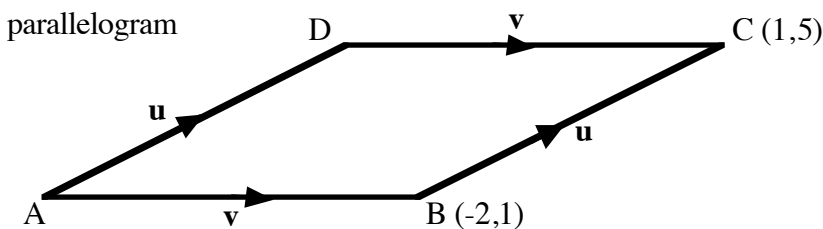
$$\overrightarrow{AD} = \overrightarrow{BC} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

same size and direction
same vector **u**
same component form

ADD/SUBTRACT

by column vectors
add or subtract components.

$$\begin{pmatrix} -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

resultant vector
a + b

by diagram
"head-to-tail" addition

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

MULTIPLY by a number

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad k\mathbf{u} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \overrightarrow{CD} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

if $\mathbf{v} = k\mathbf{u}$

    then $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel*

$\overrightarrow{CD} = 4\,\overrightarrow{AB}$

CD is parallel to AB
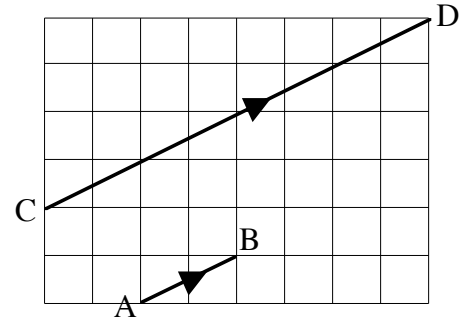
3D

Vectors in 3D operate in the same way as vectors in 2D.

Points are plotted on 3 mutually perpendicular axes.

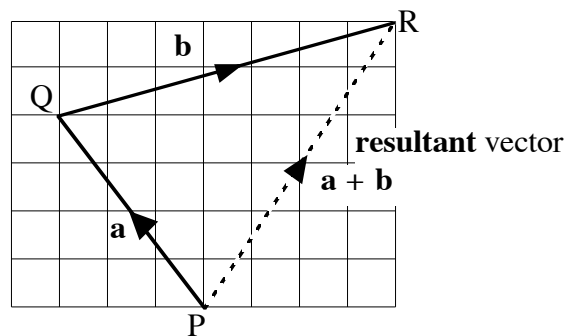$P(x,y,z)$ has **position vector** $\mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$A(5,2,7)$
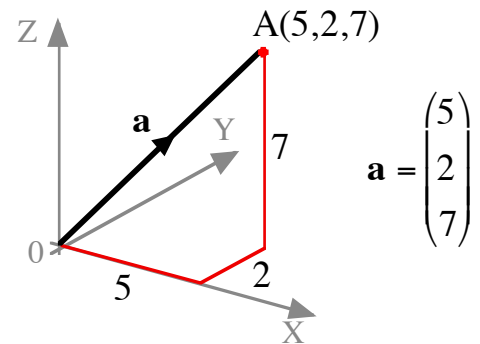
$$\mathbf{a} = \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$$

If $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ find the value of $\left| \mathbf{v} - 2\mathbf{u} \right|$.

$$\mathbf{v} - 2\mathbf{u} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ -5 \end{pmatrix}$$

$$\left| \mathbf{v} - 2\mathbf{u} \right| = \sqrt{(-3)^2 + 4^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$$

CHAPTER 20: STATISTICS

Studying statistical information, it is useful to consider: (1) typical result: **average**

(2) distribution of results: **spread**

AVERAGES:

$$mean = \frac{total\ of\ all\ results}{number\ of\ results}$$

$$median = middle\ result\ of\ the\ ordered\ results$$

$$mode = most\ frequent\ result$$

SPREAD:

Ordered results are split into 4 equal groups so each contains 25% of the results.

The **5 figure summary** identifies: L , Q_1 , Q_2 , Q_3 , H

(lowest result , 1st , 2nd and 3rd quartiles , highest result)

A **Box Plot** is a statistical diagram that displays the 5 figure summary:
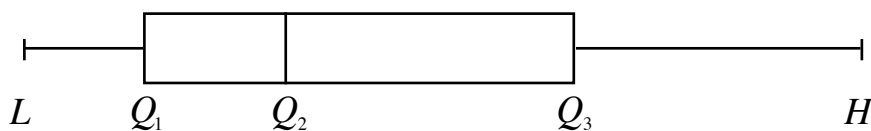
$$range,\ R = H - L$$

$$interquartile\ range,\ IQR = Q_3 - Q_1$$

$$semi-interquartile\ range,\ SIQR = \frac{Q_3 - Q_1}{2}$$

NOTE: If Q_1 , Q_2 or Q_3 fall between two results, the mean of the two results is taken.

For example,

12 ordered results: split into 4 equal groups of 3 results

$$\begin{array}{ccccccccccccc} & & & Q_1 & & & Q_2 & & & Q_3 & & & \\ 10 & 11 & 13 & | & 17 & 18 & 20 & | & 20 & 23 & 25 & | & 26 & 27 & 29 \end{array}$$
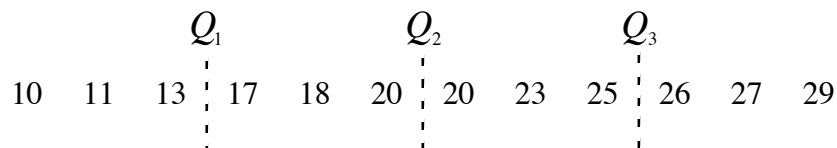
$$Q_1 = \frac{13 + 17}{2} = 15 \quad , \quad Q_2 = \frac{20 + 20}{2} = 20 \quad , \quad Q_3 = \frac{25 + 26}{2} = 25 \cdot 5$$

The pulse rates of school students were recorded in Biology class.

Pulse rates: 66, 64, 71, 56, 60, 79, 77, 75, 69, 73, 75, 62, 66, 71, 66 beats per minute.

15 ordered results:

$$\begin{matrix} & & & Q_1 & & & & Q_2 & & & & Q_3 & & & \\ 56 & 60 & 62 & \mathbf{64} & 66 & 66 & 66 & \mathbf{69} & 71 & 71 & 73 & \mathbf{75} & 75 & 77 & 79 \end{matrix}$$

5 Figure Summary:

$$L = 56 \quad , \quad Q_1 = 64 \quad , \quad Q_2 = 69 \quad , \quad Q_3 = 75 \quad , \quad H = 79$$

Box Plot:

pulse rate (beats per minute)
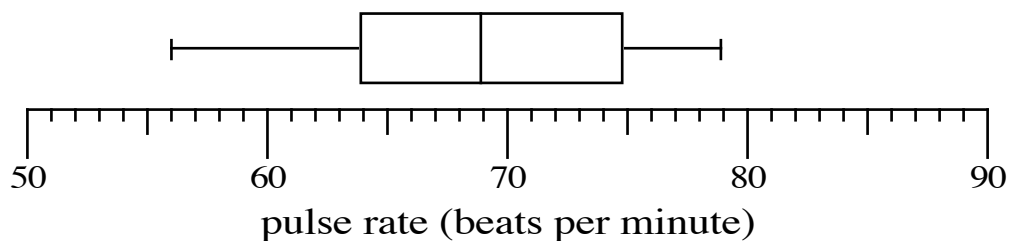
Spread:

$$R = H - L = 79 - 56 = 23$$

$$IQR = Q_3 - Q_1 = 75 - 64 = 11$$

$$SIQR = \frac{Q_3 - Q_1}{2} = \frac{75 - 64}{2} = \frac{11}{2} = 5 \cdot 5$$

Averages: $(total = 66 + 64 + 71 + ... + 66 = 1030)$

$$MEAN = \frac{1030}{15} = 68 \cdot 666... = 68 \cdot 7$$

$$(Q_2)MEDIAN = 69$$

$$MODE = 66$$

page 54

STANDARD DEVIATION

A measure of the spread of a set of data, giving a numerical value to how the data deviates from the mean. It therefore gives an indication of how good the mean is as a representitive of the data set.

Formulae:

$$\text{mean}\quad \bar{x} = \frac{\sum x}{n} \qquad \text{standard deviation}\quad s = \sqrt{\frac{\sum\left(x - \bar{x}\right)^2}{n-1}} \quad \text{or} \quad s = \sqrt{\frac{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}{n-1}}$$

Examples,

(1) High Standard Deviation: results spread out

Speed (mph)
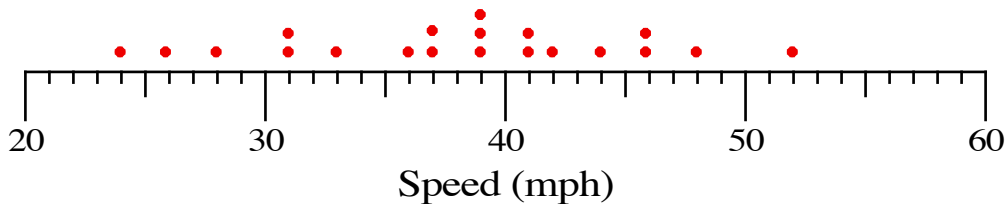
mean = 38 , standard deviation = 7·5

(2) Low Standard Deviation: results clustered around the mean
the results are more consistent

Speed (mph)

mean = 38 , standard deviation = 3·8

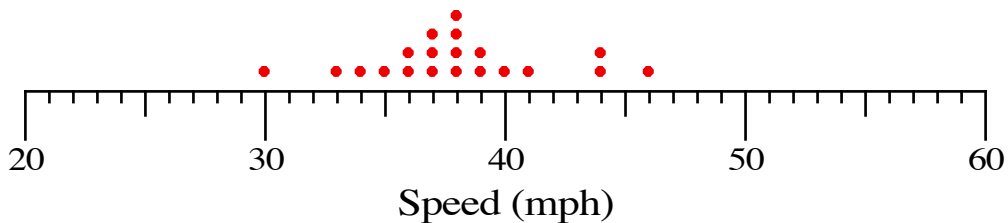The pulse rates of 8 army recruits: $61, 64, 65, 67, 70, 72, 75, 78$ beats per minute.

$$\bar{x} = \frac{\sum x}{n}$$

$$= \frac{552}{8}$$

$$= 69$$

x	$x - \bar{x}$	$(x - \bar{x})^2$
61	-8	64
64	-5	25
65	-4	16
67	-2	4
70	+1	1
72	+3	9
75	+6	36
78	+9	81
TOTALS 552		236

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$$

$$= \sqrt{\frac{236}{7}}$$

$$= 5 \cdot 806....$$

$$\approx 5 \cdot 8$$

or

$$\bar{x} = \frac{\sum x}{n}$$

$$= \frac{552}{8}$$

$$= 69$$

x	x^2
61	3721
64	4096
65	4225
67	4489
70	4900
72	5184
75	5625
78	6084
TOTALS 552	38324

$$s = \sqrt{\frac{\sum x^2 - \frac{(\sum x)^2}{n}}{n - 1}}$$

$$= \sqrt{\frac{38324 - \frac{552^2}{8}}{7}}$$

$$= \sqrt{\frac{236}{7}}$$

$$= 5 \cdot 806....$$

$$\approx 5 \cdot 8$$

PROBABILITY

The probability of an event A occuring is $\quad P(A) = \dfrac{\text{number of outcomes involving A}}{\text{total number of outcomes possible}}$

$0 \le P \le 1 \qquad P = 0 \text{ impossible} \quad , \quad P = 1 \text{ certain}$

$$P(\text{not A}) = 1 - P(A)$$

number of expected outcomes involving event A = number of trials $\times P(A)$

The experimental results will differ from the theoretical probability.

(1) In an experiment a letter is chosen at random from the word ARITHMETIC and the results recorded.

letter	frequency	relative frequency
vowel	111	$111 \div 300 = 0{\cdot}37$
consonant	189	$189 \div 300 = 0{\cdot}63$
	total = 300	total = 1

Estimate of probability, $\qquad P(vowel) = 0 \cdot 37$

(2) A letter is chosen at random from the word ARITHMETIC.

4 vowels out of 10 letters, $\qquad P(vowel) = \dfrac{4}{10} = \dfrac{2}{5} \qquad$ ie. $0 \cdot 4$

for 300 trials, $\quad$ *number of vowels expected* $= $ *number of trials* $\times P(vowel)$

$$= 300 \times 0.4$$
$$= 120$$

6 consonants out of 10 letters, $\qquad P(consonant) = \dfrac{6}{10} = \dfrac{3}{5} \qquad$ ie. $0 \cdot 6$

$P(not\ a\ vowel) = 1 - 0 \cdot 4 = 0.6$

vowel **or** consonant, $\qquad\qquad\qquad\qquad$ vowel **and** consonant,

$P(either) = \dfrac{10}{10} = 1 \quad$ certain $\qquad\qquad P(both) = \dfrac{0}{10} = 0 \quad$ impossible