WORKED EXAMPLE — TEST PAPER A

1. $f(x) = \cos 3x, \{0 \le x < 360\}$

Consider the pattern of $\cos x$ which repeats every 360°.

The period of $\cos x$ is 360°.

The period of $\cos 3x$ is $\frac{360°}{3} = 120°$.

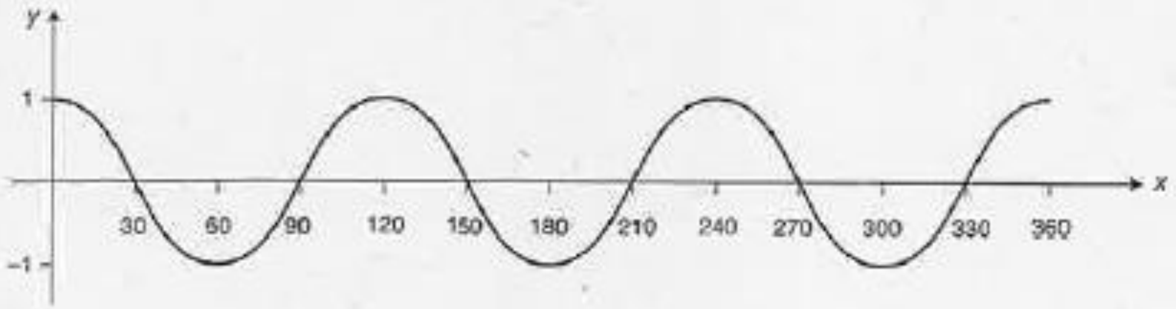
Hence the pattern repeats every 120°, almost 3 complete patterns occur within the given set $\{0 \le x < 360\}$.

x	0	30	60	90	120	150	180	210	240	270	300	330
$3x$	0	90	180	270	360	450	540	630	720	810	900	990
$\cos 3x$	1	0	−1	0	1	0	−1	0	1	0	−1	0

Points on the graph $(x, \cos 3x)$ are (0°, 1) (30°, 0) (60°, −1) (90°, 0) (120°, 1) (150°, 0), etc., as seen in the above table.

Hence the graph of $\cos 3x$ cuts the x-axis in 6 places namely (30°, 0) (90°, 0) (150°, 0) (210°, 0) (270°, 0) (330°, 0).

Graph of $\cos 3x$.



2. (a) $\text{Tan A} = \frac{5}{3}$ → sketch and label a right angled triangle

Using the theorem of Pythagoras to find the 3rd side.

$$AB^2 = AC^2 + BC^2$$
$$= 5^2 + 3^2$$
$$AB^2 = 34$$
$$AB = \sqrt{34}$$

Using the ratio of right angled triangles

$\cos A = \frac{3}{\sqrt{34}}$ $\quad \sin A = \frac{5}{\sqrt{34}}$

cos 2A Using double angle Trig. formula

$$\cos 2A = \cos A \cos A - \sin A \sin A$$
$$= \cos^2 A - \sin^2 A$$

$$\cos A = \frac{3}{\sqrt{34}}, \sin A = \frac{5}{\sqrt{34}}, \cos^2 A - \sin^2 A = \left(\frac{3}{\sqrt{34}}\right)^2 - \left(\frac{5}{\sqrt{34}}\right)^2$$
$$= \frac{9}{34} - \frac{25}{34} = -\frac{16}{34}$$
$$\cos 2A = -\frac{8}{17}$$

OR using

$$\cos^2 A - \sin^2 A$$
$$= \cos^2 A - (1 - \sin^2 A)$$
$$= 2\cos^2 A - 1$$

$\cos A = \frac{3}{\sqrt{34}}$, $= 2\left(\frac{3}{\sqrt{34}}\right)^2 - 1$

$$= 2\left(\frac{9}{34}\right) - 1 = \frac{9}{17} - \frac{17}{17}$$

$$\cos 2A = -\frac{8}{17}$$

(b) To show $\cos 2A + \sin 2A = \frac{7}{17}$

$\cos 2A = -\frac{8}{17}$ as found in part *(a)*.

By double angle formula $\sin 2A = \sin(A + A)$

$$= \sin A \cos A + \cos A \sin A$$
$$= 2 \sin A \cos A$$

$\cos A = \frac{3}{\sqrt{34}}$, $\sin A = \frac{5}{\sqrt{34}}$, 4

$$\Rightarrow 2 \sin A \cos A = 2\left(\frac{5}{\sqrt{34}}\right)\left(\frac{3}{\sqrt{34}}\right)$$
$$= \frac{30}{34}$$

$\sin 2A = \frac{15}{17}$, and from part *(a)* $\cos 2A = -\frac{8}{17}$

hence $\cos 2A + \sin 2A = -\frac{8}{17} + \frac{15}{17}$

$= \frac{7}{17}$ as given.

3. Circle has general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.
Given circle has equation $x^2 + y^2 - 8x + 11 = 0$

Hence $2g = -8$ and $2f = 0$

So centre is $(4, 0)$

$c = 11$ Radius is $\sqrt{4^2 - 11}$

$= \sqrt{5}$

Centre $(4, 0)$ radius $\sqrt{5}$ 5

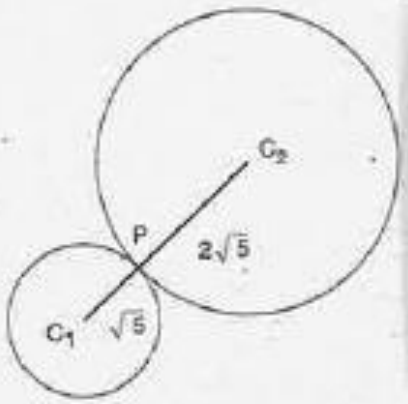
The circle with radius twice as long has radius $2\sqrt{5}$

The distance between the two centres is $\sqrt{5} + 2\sqrt{5} = 3\sqrt{5}$

Let centre of small circle $= C_1$

Let centre of large circle $= C_2$

Let point of intersection $= P(6, 1)$



$\overrightarrow{C_1P} : \overrightarrow{PC_2}$ has ratio 1 : 2

$\overrightarrow{C_1P} = p - c_1 = \begin{pmatrix}6\\1\end{pmatrix} - \begin{pmatrix}4\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

$\overrightarrow{PC_2} = 2\overrightarrow{C_1P} = 2\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$

$\overrightarrow{OC_2} = \overrightarrow{OP} + \overrightarrow{PC_2} = \begin{pmatrix}6\\1\end{pmatrix} + \begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}10\\3\end{pmatrix}$ $C_2 = (10, 3)$

Hence centre of the larger circle $= (10, 3)$

radius $= 2\sqrt{5}$

Equation is $\quad (x-10)^2 + (y-3)^2 = (2\sqrt{5})^2$

$\Leftrightarrow \quad x^2 - 20x + 100 + y^2 - 6y + 9 = 20$

$\Leftrightarrow \quad x^2 + y^2 - 20x - 6y + 89 = 0$

Alternative Method:

By using $-g = 10, -f = 3, c = 2\sqrt{5}$

in the general equation $\quad x^2 + y^2 + 2gx + 2fy + c = 0$

$\Leftrightarrow \quad x^2 + y^2 + 2(-10)x + 2(-3)y + c = 0$

$x^2 + y^2 - 20x - 6y + c = 0$

and now find c by solving $\quad r = 2\sqrt{5}$

$$r = \sqrt{g^2 + f^2 - c}$$

$$2\sqrt{5} = \sqrt{10^2 + 3^2 - c}$$

$$20 = 109 - c$$

$$\Rightarrow c = 89$$

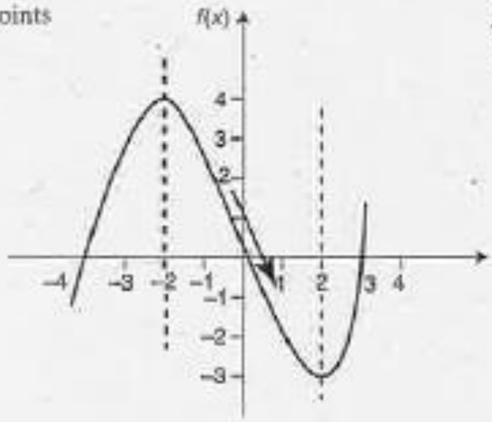
Hence equation of circle $= x^2 + y^2 - 20x - 6y + 89 = 0$

4.

$f(x) > 0, -4 < x < 0,$

$f'(x) < 0, -2 < x < 2$

$f(x)$ positive for all points above the x-axis



$f'(x) = m$, (gradient)

gradient negative between $x = -2$ and $x = 2$

$f(x) > 0$ and $f'(x) < 0$

$\Rightarrow \{x : -2 < x < 0, x \in R\}$

5. *(a)* $\quad x^2 - 6x + 4 = (x^2 - 6x) + 4$

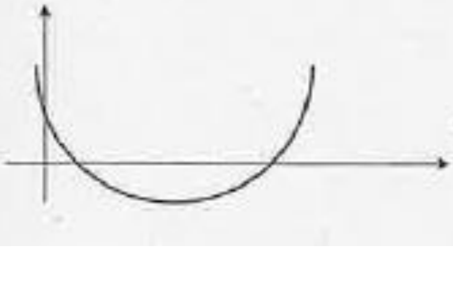
$= (x^2 - 6x + 9) + 4 - 9$

$= (x-3)^2 - 5$

Since the least value of any square number is 0, then $(x-3)^2$ has minimum value 0.
Hence the minimum value of the function is -5 when $x = 3$.
Minimum value of the function is $(3-3)^2 - 5$ and the coordinates of the minimum turning point $(3, -5)$.

(b) y-intersection $(0, 4)$, minimum turning point $(3, -5)$.

(c) Since the minimum turning point $(3, -5)$ lies below the x-axis, the curve crosses the x-axis in two places and the equation has 2 real distinct roots.



6. (a) $f(x) = x^2 - 3$ and $g(x) = 2x + 1$

$$\begin{aligned} & f(g(x)) \\ &= f(2x+1) \\ &= (2x+1)^2 - 3 \\ &= 4x^2 + 4x + 1 - 3 \\ &= 4x^2 + 4x - 2 \end{aligned}$$

$$\begin{aligned} & g(f(x)) \\ &= g(x^2 - 3) \\ &= 2(x^2 - 3) + 1 \\ & 2x^2 - 6 + 1 = 2x^2 - 5 \end{aligned}$$

$$\begin{aligned} & f(g(x)) - g(f(x)) \\ &= 4x^2 + 4x - 2 - (2x^2 - 5) \\ &= 2x^2 + 4x + 3 \end{aligned}$$

(b) If $f(g(x)) - g(f(x)) = 9$

then $2x^2 + 4x + 3 = 9$

$2x^2 + 4x - 6 = 0$; $2(x^2 + 2x - 3) = 0$; $2(x+3)(x-1) = 0$ $x = -3$ or $x = 1$

Check $f(g(-3))$ $g(-3) = 2(-3) + 1 = -5$; $f(-5) = (-5)^2 - 3 = 22$

$g(f(x)) = g(f(-3))$; $f(x) = x^2 - 3$; $f(-3) = (-3)^2 - 3 = 6$; $g(6) = 2(6) + 1 = 1$

and $22 - 13 = 9$ you could also check with $x = 1$

$f(g(1)) - g(f(1)) = 2(1)^2 + 4(1) + 3 = 9$

$f(g(-3)) - g(f(-3)) = 2(-3)^2 + 4(-3) + 3 = 9$

7. (a)

$$\begin{aligned} u_{n+1} &= 2u_n + 3 \\ u_{n+2} &= 2u_{n+1} + 3 \\ &= 2(2u_n + 3) + 3 \\ u_{n+2} &= 4u_n + 9 \end{aligned}$$

(b)

$$\begin{aligned} u_{n+1} &= 2u_n + 3 \\ u_{n+3} &= 2u_{n+2} + 3 \\ &= 2(4u_n + 9) + 3 \\ u_{n+3} &= 8u_n + 21 \\ u_{n+3} &= 53 \\ 8u_n + 21 &= 53 \\ 8u_n &= 32 \\ u_n &= 4 \end{aligned}$$

(c) Find u_{n-1}

$$\begin{aligned} u_n &= 2u_{n-1} + 3 \\ 4 &= 2u_{n-1} + 3 \end{aligned}$$

$$2u_{n-1} = 1; \quad u_{n-1} = \frac{1}{2}$$

Find u_{n+4}

$$\begin{aligned} u_{n+4} &= 2u_{n+3} + 3 \\ &= 2(53) + 3 \\ &= 109 \end{aligned}$$

8. $f(x) = (x^2 - 3x)^3$

Find $f'(x)$ by chain rule method.

$$\begin{aligned} f'(x) &= 3(x^2 - 3x)^2(2x - 3) \\ &= 3(2x - 3)(x^2 - 3x)^2 \\ f'(-1) &= 3(2(-1) - 3)((-1)^2 - 3(-1))^2 \\ &= 3(-5)(1 + 3)^2 \\ &= -15(4)^2 \\ &= -240 \\ f'(-1) &= -240 \end{aligned}$$

9. $\int_a^3 (3x^2 - 2x)dx = 20 \Rightarrow \left[\frac{3x^{2+1}}{2+1} - \frac{2x^{1+1}}{1+1}\right]_a^3 = 20$

$$= \left[\frac{3x^3}{3} - \frac{2x^2}{2}\right]_a^3 = 20$$

$$= [x^3 - x^2]_a^3 = 20$$

$$\begin{aligned} &\Rightarrow (3^3 - 3^2) - (a^3 - a^2) = 20 \\ &\Rightarrow 18 - a^3 + a^2 = 20 \\ &\Rightarrow a^2 - a^3 = 2 \end{aligned}$$

By trial and error $\Rightarrow a^2(1 - a) = 2$

assume that $a < 3$ $\Rightarrow 2^2(1 - 2) \neq 2$

$$(-1)^2(1 - (-1))$$

$$1(2) = 2$$

$$\Rightarrow a = -1$$

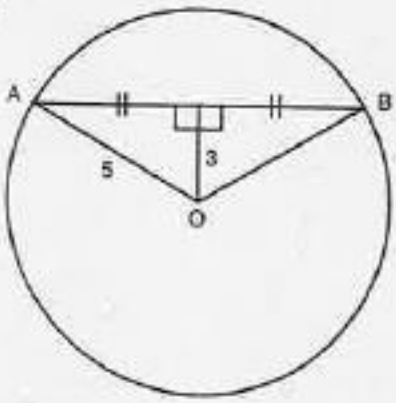
*By synthetic division using $-a^3 - a^2 = 2$ as $a^3 - a^2 + 2 = 0$

	a^3	a^2	a^1	a^0
-1	1	-1	0	2
		-1	$+2$	-2
	1	-2	$+2$	0

Since $a^3 - a^2 + 2$ is divisible by -1 then $(a + 1)$ is a factor

$\Rightarrow a = -1$

10.



Let mid-point of chord AB = P

Using the Theorem of Pythagoras on right-angled triangle $P\hat{O}B$

OB = radius = 5, | OP | = 3

$PB^2 = 5^2 - 3^2$

$PB^2 = 16 \Rightarrow PB = 4$

Using ratio of right-angled triangles $\cos P\hat{O}B = \frac{3}{5}$

$\sin P\hat{O}B = \frac{4}{5}$

To find $\sin A\hat{O}B$

$$A\hat{O}B = 2\,P\hat{O}B$$

Let $A\hat{O}B = x$, hence $P\hat{O}B = 2x$

$$\sin 2x = 2\sin x \cos x$$

$$\sin x = \frac{4}{5},\quad \cos x = \frac{3}{5}$$

$$2\sin x\cos x$$

$$= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$$

$$= 2\left(\frac{12}{25}\right)$$

$$= \frac{24}{25}$$

11. *(a)* A(3, 1, 4), B(6, 7, 10)

$$a = \begin{pmatrix}3\\1\\4\end{pmatrix},\quad b = \begin{pmatrix}6\\7\\10\end{pmatrix}$$

$\overrightarrow{AP} : \overrightarrow{PB}$

1 : 2

$$\Rightarrow\quad 2\overrightarrow{AP} = \overrightarrow{PB}$$

$$\overrightarrow{AP} = p - a \qquad \overrightarrow{PB} = b - p$$

$$\Rightarrow\quad 2\overrightarrow{AP} = 2(p-a) \qquad \overrightarrow{PB} = (b-p)$$

$$\Rightarrow 2p - 2a = b - p$$

$$3p = 2a + b$$

$$\Rightarrow\quad 3p = 2\begin{pmatrix}3\\1\\4\end{pmatrix} + \begin{pmatrix}6\\7\\10\end{pmatrix}$$

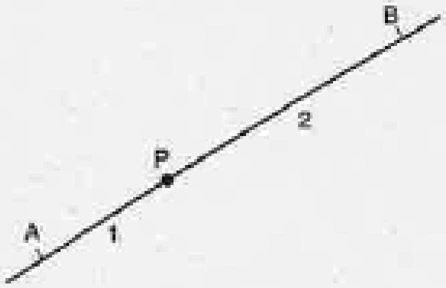
$$\begin{pmatrix}6\\2\\8\end{pmatrix} + \begin{pmatrix}6\\7\\10\end{pmatrix} = \begin{pmatrix}12\\9\\18\end{pmatrix}$$

$$3p = \begin{pmatrix}12\\9\\18\end{pmatrix} \Rightarrow p = \begin{pmatrix}4\\3\\6\end{pmatrix}$$

$p = \overrightarrow{OP}$ Hence $P = (4, 3, 6)$

Alternative Method:

This can also be solved using the section formula.



$$P = \frac{1}{3}(2a + b)$$

$$= \frac{1}{3}\left[\begin{pmatrix}6\\2\\8\end{pmatrix} + \begin{pmatrix}6\\7\\10\end{pmatrix}\right]$$

$$= \frac{1}{3}\begin{pmatrix}12\\9\\18\end{pmatrix} = (4, 3, 6)$$

(b) A(3, 1, 4), P(4, 3, 6), B(6, 7, 10)

$\Rightarrow \quad \overrightarrow{AP} = \boldsymbol{p} - \boldsymbol{a} \qquad \overrightarrow{BP} = \boldsymbol{p} - \boldsymbol{b}$

$$\boldsymbol{p} - \boldsymbol{a} = \begin{pmatrix}4\\3\\6\end{pmatrix} - \begin{pmatrix}3\\1\\4\end{pmatrix} \qquad \boldsymbol{p} - \boldsymbol{b} = \begin{pmatrix}4\\3\\6\end{pmatrix} - \begin{pmatrix}6\\7\\10\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\2\end{pmatrix} \qquad\qquad = \begin{pmatrix}-2\\-4\\-4\end{pmatrix}$$

$$\overrightarrow{AP} : \overrightarrow{BP} = 1 : -2$$

12. *(a)* Using synthetic division

	x^3	+ $6x^2$	+ $5x$	$- 12$	
$x = -3$	1	6	5	−12	
		−3	−9	−12	
	1	3	−4	0	Remainder = 0; $f(-3) = 0$

Since the remainder = 0, −3 is a root and $(x + 3)$ is a factor

$$x^3 + 6x^2 + 5x - 12 = (x + 3)(x^2 + 3x - 4)$$

By factorising the second bracket further . . . $f(x) = (x + 3)(x - 1)(x + 4)$

(b) $f(x)$ meets the x-axis when $y = 0$; $(x + 3)(x - 1)(x + 4) = 0$

$(x + 3) = 0$ or $(x - 1) = 0$ or $(x + 4) = 0$

$x = -3, x = 1, x = -4$

Coordinates are $(-3, 0)(1, 0)(-4, 0)$

$f(x)$ meets the y-axis when $x = 0$; $f(0) = -12$; coordinates are $(0, -12)$.

Hence $f(x)$ meets the axes at points $(-4, 0)$, $(-3, 0)$, $(1, 0)$, $(0, -12)$.