Recurrence Rel. Past Papers Unit 1 Outcome 4

Multiple Choice Questions

Each correct answer in this section is worth two marks.

1. A sequence is defined by the recurrence relation $u_{n+1} = \frac{1}{4}u_n + 8$ with $u_0 = 32$.

   Evaluate u_2.

   A. 10

   B. 12

   C. 16

   D. 32

Key	Outcome	Grade	Facility	Disc.	Calculator	Content	Source
B	1.4	C	0.83	0.35	NC	A11	HSN 137

$u_1 = \frac{1}{4}u_0 + 8 = \frac{1}{4} \times 32 + 8 = 8 + 8 = 16.$

$u_2 = \frac{1}{4}u_1 + 8 = \frac{1}{4} \times 16 + 8 = 4 + 8 = 12.$ Option $\boxed{B}$

2. A sequence is defined by the recurrence relation $u_{n+1} = \frac{2}{5}u_n + 6$ with $u_0 = -10$.

   What is the limit of the sequence?

   A. 10

   B. $\frac{2}{5}$

   C. $-\frac{2}{25}$

   D. -30

Key	Outcome	Grade	Facility	Disc.	Calculator	Content	Source
A	1.4	C	0.94	0.14	NC	A13	HSN 088

A limit exists since $-1 < \frac{2}{5} < 1$.

Method 1 $l = \dfrac{b}{1-a}$ where $a = \frac{2}{5}$, $b = 6$

$$= \frac{6}{1-\frac{2}{5}}$$

$$= \frac{6}{3/5}$$

$$= 10.$$

Method 2 As $n \to \infty$, $u_{n+1} = u_n = l$.

$$l = \frac{2}{5}l + 6$$

$$\frac{3}{5}l = 6$$

$$l = 10.$$

Option $\boxed{A}$

[END OF MULTIPLE CHOICE QUESTIONS]

Written Questions

[SQA] 3. Two sequences are defined by these recurrence relations:

$u_{n+1} = 3u_n - 0.4$ with $u_0 = 1$, $\quad v_{n+1} = 0.3v_n + 4$ with $v_0 = 1$.

(*a*) Explain why only one of these sequences approaches a limit as $n \to \infty$. **1**

(*b*) Find algebraically the exact value of the limit. **2**

(*c*) For the other sequence, find

(i) the smallest value of n for which the n^{th} term exceeds 1000, and

(ii) the value of that term. **2**

part	marks	Unit	non-calc C	non-calc A/B	calc C	calc A/B	calc neut C	calc neut A/B	Content Reference: Main	Additional	1.4
(a)	1	1.4			1				1.4.4		Source
(b)	2	1.4			2				1.4.5		1998 P1 qu.8
(c)	2	1.4				2			1.4.3		

•1 Only V_n has a limit because $-1 < 0.3 < 1$

•2 e.g. use $L = aL + b$

•3 $L = \frac{40}{7}$

•4 evaluate enough terms to exceed 1000

•5 $u_7 = 1749.8$

[SQA] 4. A sequence is defined by the recurrence relation $u_n = 0{\cdot}9u_{n-1} + 2$, $u_1 = 3$.

(*a*) Calculate the value of u_2. **1**

(*b*) What is the smallest value of n for which $u_n > 10$? **1**

(*c*) Find the limit of this sequence as $n \to \infty$. **2**

part	marks	Unit	non-calc C	non-calc A/B	calc C	calc A/B	calc neut C	calc neut A/B	Content Reference: Main	Additional	1.4
(a)	1	1.4			1				1.4.2		Source
(b)	1	1.4			1				1.4.3		1994 P1 qu.9
(c)	2	1.4			2				1.4.5		

•1 $4{\cdot}7$

•2 7

•3 $l = 0{\cdot}9l + 2$ OR •3 $l = \text{“}\frac{b}{1-a}\text{”} = \frac{2}{1-0{\cdot}9}$

•4 20 •4 20

[SQA] 5. A sequence is defined by the recurrence relation $u_{n+1} = 0{\cdot}3u_n + 5$ with first term u_1.

(*a*) Explain why this sequence has a limit as n tends to infinity. **1**

(*b*) Find the **exact** value of this limit. **2**

part	marks	Unit	non-calc C	non-calc A/B	calc C	calc A/B	calc neut C	calc neut A/B	Content Reference: Main	Additional	1.4
(a)	1	1.4					1		1.4.4		Source
(b)	2	1.4					2		1.4.5		1996 P1 qu.11

•1 $-1 < 0{\cdot}3 < 1$

•2 $L = 0{\cdot}3L + 5$

or $L = \frac{b}{1-a} = \frac{5}{1-0{\cdot}3}$

•3 $L = \frac{50}{7}$

[SQA] 6. Two sequences are generated by the recurrence relations $u_{n+1} = au_n + 10$ and $v_{n+1} = a^2v_n + 16$.

The two sequences approach the same limit as $n \to \infty$.

Determine the value of a and evaluate the limit. **5**

Part	Marks	Level	Calc.	Content	Answer	U1 OC4
	4	C	NC	A13	$a = \frac{3}{5}, L = 25$	2000 P1 Q5
	1	A/B	NC	A12		

•1 ss: know how to find limit •2 pd: process •3 pd: process •4 ic: interpret coeff. of u_n •5 pd: process	•1 $L = aL + 10$ **or** $L = a^2L + 16$ **or** $L = \frac{b}{1-a}$ •2 $L = \frac{10}{1-a}$ **or** $L = \frac{16}{1-a^2}$ •3 $\frac{10}{1-a}$ **or** $\frac{16}{1-a^2}$ •4 $10a^2 - 16a + 6 = 0$ •5 $a = \frac{3}{5}$ **and** $L = 25$

[SQA] 7. Two sequences are defined by the recurrence relations

$$u_{n+1} = 0 \cdot 2u_n + p, \quad u_0 = 1 \quad \text{and}$$
$$v_{n+1} = 0 \cdot 6v_n + q, \quad v_0 = 1.$$

If both sequences have the same limit, express p in terms of q. 3

part	marks	Unit	non-calc		calc		calc neut		Content Reference :		1.4
			C	A/B	C	A/B	C	A/B	Main	Additional	Source
.	3	1.4						3	1.4.5		1999 P1 qu.18

- •[1] "$L = 0.2L + p, \quad L = 0.6L + q$" *or use* "$l = \frac{b}{1-a}$"
- •[2] $\frac{p}{0.8}$ *and* $\frac{q}{0.4}$
- •[3] $p = \dfrac{0.8q}{0.4}$ or equivalent expression for p

[SQA] 8. On the day of his thirteenth birthday, a boy is given a sum of money to invest and instructions not to withdraw any money until after his eighteenth birthday. The money is invested and compound interest of 9% per annum is added each following birthday. By what percentage will the investment have increased when he withdraws his money just after his eighteenth birthday? 3

part	marks	Unit	non-calc		calc		calc neut		Content Reference :		1.4
			C	A/B	C	A/B	C	A/B	Main	Additional	Source
.	3	1.4			3				1.4.3		1991 P1 qu.11

- •[1] 1.09
- •[2] for using $(\ldots)^5$
- •[3] *approx.* 54%

[SQA] 9. (*a*) At 12 noon a hospital patient is given a pill containing 50 units of antibiotic.
By 1 pm the number of units in the patient's body has dropped by 12%.
By 2 pm a further 12% of the units remaining in the body at 1 pm is lost.
If this fall-off rate is maintained, find the number of units of antibiotic remaining at 6 pm. **(4)**

(*b*) A doctor considers prescribing a course of treatment which involves a patient taking one of these pills every 6 hours over a long period of time.
The doctor knows that mopre than 100 units of this antibiotic in the body is regarded as too dangerous.
Should the doctor prescribe this course of treatment?
Give reasons for your answer. **(6)**

part	marks	Unit	non-calc C	non-calc A/B	calc C	calc A/B	calc neut C	calc neut A/B	Content Reference: Main	Additional	1.4
(*a*)	4	1.4			4				1.4.1		Source
(*b*)	6	1.4			4	2			1.4.3, 1.4.5		1991 Paper 2 Qu. 9

(*a*) •1 use 0.88 or 88%
•2 $n = 6$
•3 $u_6 = 50 \times 0.88^6$
•4 23.22

(*b*) •5 adding 50
•6 $u_{n+1} = 0.88^6 u_n + 50$
•7 $-1 < 0.88^6$ (*or* $0.4644) < 1$ so limit exists
•8 $L = \frac{50}{1-0.88^6}$
•9 93.4
•10 $93.4 < 100$ so safe to continue

[SQA] 10. The extract below is taken from the "OIL RIG NEWS".

RARE ILLNESS STRIKES RIG

Storm prevents delivery of medicine

By noon on Tuesday 20th December 1988 50 of our oil rig personnel were laid low by a mystery illness.

Our resident medical officer is expressing concern because the number of personnel affected is increasing each day by 8% of the previous day's total.

(*a*) If the daily rate of increase remained at 8% of the previous day's total, how many personnel were affected by noon on Sunday 25th December 1988? **(3)**

(*b*) An improvement in the weather conditions allowed a team of medics to fly out to the rig on the morning of Tuesday 27th December 1988.
At noon on that Tuesday, all personnel were innoculated and no new cases of the illness arose. Within the next 24 hours, 21% of those who had been affected had recovered.
If the daily rate of recovery of 21% of the previous day's total was maintained, how many personnel were still affected by the illness at noon on Saturday 31st December 1988? **(5)**

part	marks	Unit	non-calc C	non-calc A/B	calc C	calc A/B	calc neut C	calc neut A/B	Content Reference: Main	Additional	1.4
(*a*)	3	1.4			3				1.4.2		Source
(*b*)	5	1.4			5				1.4.2		1990 Paper 2 Qu. 3

(*a*)
•[1] $u_n = 1.08^n u_0$
•[2] $u_5 = 1.08^5 \times 50$
•[3] 73 or 74

(*b*)
•[4] $u_7 = 1.08^7 \times 50$
•[5] $u_7 = 85$ or 86
•[6] $v_n = 0.79^n v_0$
•[7] $v_4 = 33$ or 34
•[8] for consistent rounding

[SQA] 11. Secret Agent 004 has been captured and his captors are giving him a 25 milligram dose of a truth serum every 4 hours. 15% of the truth serum present in his body is lost every hour.

(*a*) Calculate how many milligrams of serum remain in his body after 4 hours (that is immediately **before** the second dose is given). **(3)**

(*b*) It is known that the level of serum in the body has to be continuously above 20 milligrams before the victim starts to confess. Find how many doses are needed before the captors should begin their interrogation. **(3)**

(*c*) Let u_n be the amount of serum (in milligrams) in his body just **after** his n^{th} dose. Show that $u_{n+1} = 0 \cdot 522u_n + 25$. **(1)**

(*d*) It is also known that 55 milligrams of this serum in the body will prove fatal, and the captors wish to keep Agent 004 alive. Is there any maximum length of time for which they can continue to administer this serum and still keep him alive ? **(4)**

part	marks	Unit	non-calc C	non-calc A/B	calc C	calc A/B	calc neut C	calc neut A/B	Content Reference: Main	Additional	1.4
(*a*)	3	1.4			3				1.4.1		Source
(*b*)	3	1.4			3				1.4.1		**1993 Paper 2**
(*c*)	1	1.4			1				1.4.3		**Qu.8**
(*d*)	4	1.4			3	1			1.4.4,	1.4.5	

(*a*)	•1	strategy for each hour (e.g. using 0·85)
	•2	using strategy 4 times (e.g. $(0{\cdot}85^4)$)
	•3	13·05
(*b*)	•4	apply a correct dose strategy
	•5	a relevant sequence e.g. 13·05, 19·86, 23·4, or 25, 38·05, 44·9, 48·4
	•6	3 doses
(*c*)	•7	valid explanation i.e. $(0 \cdot 85)^4 = 0 \cdot 522$ explicitly stated
(*d*)	•8	statement that limit exists because $(0 \cdot 85)^4 < 1$
	•9	$\therefore l = 0 \cdot 522l + 25$ or using $l = \frac{b}{1-a}$
	•10	$l = 52{\cdot}3$
	•11	$52{\cdot}3 < 55$ so no maximum length of time

[SQA] 12. The sum of £1000 is placed in an investment account on January 1st and, thereafter, £100 is placed in the account on the first day of each month.

- Interest at the rate of 0·5% per month is credited to the account on the last day of each month.
- This interest is calculated on the amount in the account on the first day of the month.

(*a*) How much is in the account on June 30th ? **(4)**

(*b*) On what date does the account first exceed £2000? **(2)**

(*c*) Find a recurrence relation which describes the amount in the account, explaining your notation carefully. **(3)**

part	marks	Unit	non-calc C	non-calc A/B	calc C	calc A/B	calc neut C	calc neut A/B	Content Reference: Main	Additional	1.4
(*a*)	4	1.4			4				1.4.1		Source
(*b*)	2	1.4			2				1.4.1		1997 Paper 2
(*c*)	3	1.4			3				1.4.3		Qu.3

(a)
- •[1] 1.005
- •[2] £1000 + interest = £1005
- •[3] £1005 + £100 + interest = £1110.525
- •[4] £1537.93

(b)
- •[5] complete another month
- •[6] £2073.94 on Nov.1st

(c)
- •[7] $u_{n+1} = 1.005u_n + 100$
- •[8] u_n = amount on 1st day of each month
- •[9] $u_0 = 1000$ (on 1st January)

[SQA] 13. A gardener feeds her trees weekly with "Bioforce, the wonder plant food". It is known that in a week the amount of plant food in the tree falls by about 25%.

(*a*) The trees contain no Bioforce initially and the gardener applies 1g of Bioforce to each tree every Saturday. Bioforce is only effective when there is continuously more than 2g of it in the tree. Calculate how many weekly feeds will be necessary before the Bioforce becomes effective. **(3)**

(*b*) (i) Write down a recurrence relation for the amount of plant food in the tree immediately after feeding. **(1)**

(ii) If the level of Bioforce in the tree exceeds 5g, it will cause leaf burn. Is it safe to continue feeding the trees at this rate indefinitely? **(4)**

part	marks	Unit	non-calc C	non-calc A/B	calc C	calc A/B	calc neut C	calc neut A/B	Content Reference: Main	Additional	1.4
(*a*)	3	1.4			3				1.4.1		Source
(*b*)	1	1.4			1				1.4.3		**1998 Paper 2**
(*c*)	4	1.4			4				1.4.4, 1.4.5		**Qu. 8**

(*a*) •[1] 75% or equivalent

•[2] 0.75, 1.31 **and** 1.73

•[3] 2.05 **and** "after fourth feed"

(*b*) •[4] $u_{n+1} = 0.75u_n + 1$

(*c*) •[5] $-1 < 0.75 < 1$ so sequence has a limit

•[6] *e.g.* $L = 0.75L + 1$

•[7] $L = 4$

•[8] Safe to continue

[SQA] 14. Some environmentalists are concerned that the presence of chemical nitrates in drinking water presents a threat to health.

The World Health Organisation recommends an upper limit of 50 milligrams per litre (mg/l) for nitrates in drinking water, although it regards levels up to 100 mg/l as safe.

A sub-committee of a Local Water Authority is considering a proposal affecting a small loch which supplies a nearby town with drinking water. The proposal is that a local factory be permitted to make a once-a-week discharge of effluent into the loch, provided that a cleaning treatment of the loch is carried out before each discharge of effluent.

The Water Engineer has presented the following data:

1. The present nitrate level in the loch is 20 mg/l.
2. The cleaning treatment removes 55% of the nitrates from the loch.
3. Each discharge of effluent will result in an addition of 26 mg/l to the nitrate presence in the loch.

and advises the sub-committee that the proposal presents no long-term danger from nitrates to the drinking water supply.

(*a*) Show the calculations you would use to check the engineer's advice. **(5)**

(*b*) Is the engineer's advice acceptable? **(1)**

part	marks	Unit	non-calc C	non-calc A/B	calc C	calc A/B	calc neut C	calc neut A/B	Content Reference: Main, Additional	1.4
(*a*)	5	1.4			5				1.4.1, 1.4.5	Source
(*b*)	1	0.1			1				0.1	**1989 Paper 2 Qu. 6**

(*a*) •[1] $u_0 = 20$

•[2] $u_1 = 35$

•[3] three further values eg 41.75, 44.78, 46.15

•[4] 46.76, 47.04, 47.17 looks like approaching a limit

•[5] five more lead to 47.27'something' $\Rightarrow$ limit = 47.27

(*b*) •[6] $47.27 < 50$ so level safe

[SQA] 15. On the first day of March, a bank loans a man £2500 at a fixed rate of interest of 1.5% per month. This interest is added on the last day of each month and is calculated on the amount due on the first day of the month. He agrees to make repayments on the first day of each subsequent month. Each repayment is £300 except for the smaller final amount which will pay off the loan.

(*a*) The amount that he owes at the start of each month is taken to be the amount still owing just after the monthly repayment has been made.

Let u_n and u_{n+1} represent the amounts that he owes at the start of two successive months. Write down a recurrence relation involving u_{n+1} and u_n. **2**

(*b*) Find the date and the amount of the final payment. **4**

Part	Marks	Level	Calc.	Content	Answer	U1 OC4
(*a*)	2	C	CN	A10, A14	$u_{n+1} = 1{\cdot}015u_n - 300, u_0 = 2500$	2001 P2 Q3
(*b*)	4	C	CR	A11, A14	1 December, £290·68	

•1 ic: interpret 1·5%	•1 1·015 *stated or implied by the start of (b)*
•2 ic: state the recurrence relation	•2 $u_{n+1} = 1{\cdot}015u_n - 300$ and initial value (e.g. $u_0 = 2500$) *stated or implied by the start of (b)*
•3 ss: use recurrence relation	•3 u_1 i.e. £2237·50
•4 pd: process	•4 u_2 **and** u_3 i.e. £1971·06, £1700·63
•5 ic: start final date	•5 £286·38
•6 pd: process final payment	•6 £290·68 for December payment

[SQA] 16. A man decides to plant a number of fast-growing trees as a boundary between his property and the property of his next door neighbour. He has been warned, however, by the local garden centre that, during any year, the trees are expected to increase in height by 0·5 metres. In response to this warning he decides to trim 20% off the height of the trees at the start of any year.

(*a*) If he adopts the "20% pruning policy", to what height will he expect the trees to grow in the long run? **3**

(*b*) His neighbour is concerned that the trees are growing at an alarming rate and wants assurances that the trees will grow no taller than 2 metres. What is the minimum percentage that the trees will need to be trimmed each year so as to meet this condition. **3**

Part	Marks	Level	Calc.	Content	Answer	U1 OC4
(*a*)	3	C	CN	A13, A14	2·5 metres	2002 P2 Q4
(*b*)	3	C	CN	A12, A13	trim 25%	

•[1] ic: interpret the decay factor	•[1] 0·8 *stated or implied*
•[2] ss: strategy for limit	•[2] e.g. $l = 0{\cdot}8l + 0{\cdot}5$ **or** $l = \frac{0\cdot5}{1-0\cdot8}$
•[3] pd: process limit	•[3] $-1 < 0{\cdot}8 < 1$ so $l = 2{\cdot}5$ metres
•[4] ss: reverse strategy for limit	•[4] $2 = 2m + 0{\cdot}5$
•[5] pd: process	•[5] $m = 0{\cdot}75$
•[6] ic: interpret scale factor	•[6] trim 25%

[SQA] 17. Biologists calculate that when the concentration of a particular chemical in a sea loch reaches 5 milligrams per litre (mg/l) the level of pollution endangers the life of the fish.
A factory wishes to release waste containing this chemical into the loch. It is claimed that the discharge will not endanger the fish.
The Local Authority is supplied with the following information:

1. The loch contains none of this chemical at present.
2. The factory manager has applied to discharge effluent once per week which will result in an increase in concentration of 2.5 mg/l of the chemical in the loch.
3. The natural tidal action will remove 40% of the chemical from the loch every week.

(*a*) Show that this level of discharge would result in fish being endangered. **(3)**

When this result is announced, the company agrees to install a cleaning process that reduces the concentration of chemical released into the loch by 30%.

(*b*) Show the calculations you would use to check this revised application.
Should the Local Authority grant permission ? **(5)**

part	marks	Unit	non-calc C	non-calc A/B	calc C	calc A/B	calc neut C	calc neut A/B	Content Reference: Main	Additional	1.4
(*a*)	3	1.4			3				1.4.3		Source
(*b*)	5	1.4			5				1.4.3,	1.4.5	1992 Paper 2 Qu.3

(*a*) •¹ 0.6 stated/implied
•² $u_{n+1} = 0.6u_n + 2.5$
•³ communication: ie 6.25 $\Rightarrow$ danger

(*b*) •⁴ $0.7 \times 2.5 = 1.75$
•⁵ 2.8, 3.43, 3.808
•⁶ $u_{n+1} = 0.6u_n + 1.75$
•⁷ limit = 4.375
•⁸ communication: ie 4.375 $\Rightarrow$ allow/disallow

[SQA] 18. Trees are sprayed weekly with the pesticide, KILLPEST, whose manufacturers claim it will destroy 65% of all pests. Between the weekly sprayings it is estimated that 500 new pests invade the trees.
A new pesticide, PESTKILL, comes onto the market. The manufacturers claim that it will destroy 85% of existing pests but it is estimated that 650 new pests per week will invade the trees.
Which pesticide will be more effective in the long term ? **(7)**

part	marks	Unit	non-calc		calc		calc neut		Content Reference :		1.4
			C	A/B	C	A/B	C	A/B	Main	Additional	Source
-	7	1.4			7				1.4.3,	1.4.4, 1.4.5	**1995 Paper 2 Qu.3**

(-) $\bullet^1$ $0\cdot35$ stated or implied

$\bullet^2$ $0\cdot35u_n + 500$

$\bullet^3$ $0\cdot15$ stated or implied

$\bullet^4$ $0\cdot15u_n + 650$

$\bullet^5$ $l = al + b\ldots\ldots\ldots$ *or* limit $= \frac{b}{1-a}\ldots\ldots\ldots$

$\bullet^6$ limits $= 769$ and 765

$\bullet^7$ Limits are valid since $|a| < 1$ in both cases
and Pestkill is more effective

[END OF WRITTEN QUESTIONS]